Freie Universität Berlin

Étale and Pro-Étale Fundamental Groups

Master's Thesis

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All these tempests that fall upon us
are signs that fair weather is coming shortly,
and that things will go well with us,
for it is impossible for good or evil to last forever;
and hence it follows that
the evil having lasted so long,
the good must be now nigh at hand.

-DON QUIXOTE

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1. Introduction

In this masters thesis we look at fundamental groups and Galois theory in various guises. Galois introduced the groups now named after him in the 1830's (although in a different manner than we know them today, since abstract group theory was still years in the future) and they quickly became a subject of intense study and a catalyst for the abstract study of groups and other algebraic structures. In the 1890's Poincaré defined the fundamental group of a topological space and in doing so initiated the study of algebraic topology.

Soon the similarities between the theories became apparent and in the 1960's Grothendieck introduced the étale fundamental group and showed how in some sense it encompasses (parts of) the theory of the fundamental groups and Galois theory. More precisely, we can view absolute Galois groups as étale fundamental groups of points, and if we are working with schemes of finite type over $\mathbb{C}$ we can associate with it a complex-analytic space and the étale fundamental group of our scheme is the profinite completion of the topological fundamental group of this complex-analytic space.

There are however some problems with the étale fundamental group. In topology one has an equivalence between the category of locally constant sheaves of complex vector spaces and the category of finite dimensional complex representation of the fundamental group. In the setting of algebraic geometry one would hope that this result could be extended to the étale fundamental group, where the field $\mathbb{C}$ is replaced by $\mathbb{Q}_l$ or $\bar{\mathbb{Q}}_l$. This is however not so and we see an example where this fails.

This led Grothendieck and his school to look at lisse sheaves which are certain projective systems of étale sheaves. Replacing our naïve local systems by these lisse sheaves allows us to recover this equivalence.

Theses lisse sheaves are at the foundation of the construction of l-adic cohomology, which has been a very successful theory and led to many breakthroughs. However l-adic cohomology is not realized as sheaf cohomology of a sheaf on the étale site, but as an inverse limit of étale cohomology groups with torsion coefficients. This is one of the motivation for Bhatt and Scholze in 2010's to construct the pro-étale site. They are able to recover l-adic cohomology as sheaf cohomology of a sheaf on the pro-étale site. In particular they recover the equivalence between locally constant sheaves of $\mathbb{Q}_l$-vector spaces, and the continuous $\mathbb{Q}_l$-representation of the pro-étale fundamental group. Furthermore they show that the étale fundamental group can be recovered as the profinite completion of the pro-étale fundamental group in general, and that they actually agree in many cases.

In this thesis we aim to describe these results of Bhatt and Scholze for the fundamental group. This thesis is expository and contains no new results or ideas.

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2. Galois theory

We start by recalling the fundamental theorem of finite Galois theory. This is a neat and precise result that one would hope would extend naïvely to infinite Galois extensions. However as early as 1901 Dedekind found a counterexample.

To understand how we can remidy this situation we recall the definition of profinite groups, a certain class of topological groups that are in a sense determined by their finite quotients, and present Krull's theorem on infinite Galois extensions.

Finally we introduce finite dimensional étale k-algebras and discuss a generalization of Krull's theorem due to Grothendieck.

2.1. **Krull's theorem on infinite Galois extensions.** Finite Galois theory is classical and can be found in any textbook on Galois theory. For infinite Galois theory, and the reformulation of Grothendieck we recommend Szamuely [2009].

Recall firstly that given a field extension L/k we can assign to it the Galois group $\mathrm{Gal}(L/k)$ of all automorphisms of L that fix k pointwise. In general the action of $\mathrm{Gal}(L/k)$ may fix a larger subfield of L than k.

Definition 2.1. A field extension L/k is called *Galois* if the field that is fixed by the action of $\mathrm{Gal}(L)$ is precisely k.

Theorem 2.2 (Fundamental Theorem of Finite Galois Theory)**.** *Let K be a finite Galois extension of k and let G denote the Galois group $\mathrm{Gal}(K/k)$. Then there is a $1 \leftrightarrow 1$ correspondence between subextensions of K and subgroups of G. This correspondence is given by*

$$F \mapsto \mathrm{Gal}(K/F)$$
$$H \mapsto K^H$$

Moreover F/k is Galois if and only if $H := \mathrm{Gal}(K/F)$ is a normal subgroup of G, in which case

$$\mathrm{Gal}(F/k) \cong G/H$$

Definition 2.3. A *profinite* group is a topological group that is the inverse limit of a system of finite groups, each endowed with the discrete topology. For a prime number p, a *pro-p group* is an inverse limit of a system of finite p-groups.

Example 2.4. Let p be a fixed prime number. For each $n \in \mathbb{Z}_{>0}$ we consider the group $G_n := \mathbb{Z}/p^n\mathbb{Z}$. We order them by saying that $G_n \preceq G_m$ if and only if $n \leq m$. This is precisely when we have canonical quotient maps

$$\phi_n^m : G_m := \mathbb{Z}/p^m\mathbb{Z} \to \mathbb{Z}/p^n\mathbb{Z} =: G_n$$

The groups G_n along with these maps, form an inverse system. The inverse limit is the additive group of *p-adic integers*.

One can easily see that profinite groups are Hausdorff as they are inverse limits of Hausdorff spaces. They are also easily seen to be totally disconnected and applying Tychonoff's theorem shows that they are compact.

We have the following classical proposition that tells us that in fact, these three properties are equivalent to profiniteness.

Proposition 2.5. *A topological group G is profinite if and only if it is Hausdorff, totally disconnected, and compact.*

Example 2.6. Consider the groups $U_n = \mathbb{Z}/n\mathbb{Z}$ for all $n \in \mathbb{Z}_{>0}$. We order them by declaring

$$U_n \preceq U_m \Leftrightarrow U_m \subseteq U_n$$

i.e.

$$U_n \preceq U_m \Leftrightarrow n|m$$

For each $n|m$ we have a canonical map

$$\phi_n^m : U_m = \mathbb{Z}/m\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z} = U_n$$

and the groups U_n together with these maps form an inverse system. The inverse limit, an important group in algebraic number theory among other fields, is denoted by $\widehat{\mathbb{Z}}$ and is called the *profinite completion of the integers*, or *the ring of profinite integers*.

The last example is a special case of a construction we shall need later in our discussions of the étale fundamental groups.

Definition 2.7. Let G be a group. One can construct an inverse system from the set of its finite quotients. Let $(N_\alpha)_{\alpha\in I}$ be the indexed family of all normal subgroups of finite index ordered by reverse inclusion, i.e.

$$N_\alpha \preceq N_\beta$$
$$\Leftrightarrow$$
$$N_\beta \subseteq N_\alpha$$

We have canonical quotient maps for each $N_\alpha \preceq N_\beta$

$$\phi_\alpha^\beta : G/N_\beta \to G/N_\alpha$$

Now the quotients and these maps form an inverse system whose inverse limit is called the *profinite completion* of G, denoted by $\widehat{G}$.

Proposition 2.8. *Let K/k be a Galois extension. The Galois groups of finite Galois subextensions of K/k with the canonical restriction maps*

$$\phi_{ML} : \mathrm{Gal}(M/k) \to \mathrm{Gal}(L/k)$$

for $k \subseteq L \subseteq M \subseteq K$, form an inverse system. The inverse limit of this system is isomorphic to $\mathrm{Gal}(K/k)$*. In particular* $\mathrm{Gal}(K/k)$ *is profinite.*

We will need the following lemma for the interesting example of Dedekind, 2.10.

Lemma 2.9. *Let k be a field and fix a seperable closure k_s of it. Then a subextension F/k of k_s/k is Galois if and only if for any $\sigma \in \mathrm{Gal}(k)$ we have $\sigma(F) \subset F$.*

Example 2.10 (Dedekind [1931])**.** Consider the extension $F/\mathbb{Q}$ where $F = \mathbb{Q}(\mu_{p^\infty})$ is obtained by adjoining all p-th power roots of unity for a fixed odd prime number p. Notice that any element $\sigma \in \mathrm{Gal}(\mathbb{Q})$ sends any n-th root of unity to an n-th root of unity. Hence $\sigma(F) \subseteq F$ and the above lemma 2.9 tells us $F/\mathbb{Q}$ is Galois. Now any $\sigma \in \mathrm{Gal}(\mathbb{Q})$ is uniquely determined by where it sends all p^n-th roots of unity, for all $n \in \mathbb{N}$. The finite Galois extension $\mathbb{Q}(\mu_{p^n})$, where

$$\mu_{p^n} = e^{2\pi i/p^n}$$

has the Galois group

$$\mathrm{Gal}(\mathbb{Q}(\mu_{p^n})) \cong (\mathbb{Z}/p^n\mathbb{Z})^\times$$

The elements in $\mathrm{Gal}(\mathbb{Q}(\mu_{p^n}))$ are given by

$$\mu_{p^n} \mapsto \mu_{p^n}^{r_n}$$

for some $r_n \in (\mathbb{Z}/p^n\mathbb{Z})^\times$. If we start with an element $\pi \in \mathrm{Gal}(F/\mathbb{Q})$ we can condsider it as an element of $\mathrm{Gal}(\mathbb{Q}(\mu_{p^n}))$ for all n, by restriction. This π acts by raising to the power r_n on the p^n-th roots. But notice that

$$\mu_{p^{n+1}}^{p^n} = \mu_{p^n}$$

and therefore

$$\begin{aligned}\mu_{p^n}^{r_n} &= \pi(\mu_{p^n})\\ &= \pi(\mu_{p^{n+1}}^{p^n})\\ &= \pi(\mu_{p^{n+1}})^{p^n}\\ &= (\mu_{p^{n+1}}^{r_{n+1}})^{p^n}\\ &= (\mu_{p^{n+1}}^{p^n})^{r_{n+1}}\\ &= \mu_{p^n}^{r_{n+1}}\end{aligned}$$

So we then have the compatibility condition $r_{n+1} \equiv r_n \mod p^n$ and we can consider the inverse limits. We get:

$$\mathbb{Z}_p^\times \cong \varprojlim (\mathbb{Z}/p^n\mathbb{Z})^\times \cong \varprojlim \mathrm{Gal}(\mathbb{Q}(\mu_{p^n})/\mathbb{Q}) \cong \mathrm{Gal}(F/\mathbb{Q})$$

We have a direct product decomposition

$$(\mathbb{Z}/p^n\mathbb{Z})^\times \cong (\mathbb{Z}/p\mathbb{Z})^\times \times ((1+p\mathbb{Z})/(1+p^n\mathbb{Z}))$$

which induces in the limit a decomposition of the p-adic units into a direct product

$$\mathbb{Z}_p^\times \cong (\mathbb{Z}/p\mathbb{Z})^\times \times (1+p\mathbb{Z}_p)$$

Consider then the subgroups of $\mathbb{Z}_p^\times$ of the form

$$H_e \times G_m$$

where $e|(p-1), m \in \mathbb{N}\cup\{\infty\}$ and H_e is the unique subgroup of $(\mathbb{Z}/p\mathbb{Z})^\times$ of index e, $G_\infty = \{1\}$ and

$$G_m = 1 + p^{m+1}\mathbb{Z}_p$$

The Galois correspondence associates with these subgroups their fixed fields and we obtain the subfields of F given as composites

$$\mathbb{F}_{e,m} = \mathbb{Q}_e \cdot \mathbb{K}^{m+1}$$

where as before $e|(p-1), m \in \mathbb{N} \cup \{\infty\}$ and now $\mathbb{Q}_e/\mathbb{Q}$ is the unique subextension of $\mathbb{Q}(\mu_p)/\mathbb{Q}$ of degree e,

$$\mathbb{K}^\infty := F^{(\mathbb{Z}/p\mathbb{Z})^\times}$$

and

$$\mathbb{K}^{m+1} := \mathbb{K}^\infty \cap \mathbb{Q}(\mu_{p^{m+1}})$$

This list of subfields exhausts all the subfields of F, but there are othere subgroups of $\mathbb{Z}_p^\times$. Therefore the Galois correspondence sending subgroups of the Galois group to the fixed fields *is not injective*!

This preceding example shows that the Fundamental Theorem of Finite Galois theory does not extend naively to the infinite case.

However, one can see that the subgroups listed there are precisely the *closed subgroups* of $\mathrm{Gal}(F/\mathbb{Q})$. That is precisely the crux of the matter.

Theorem 2.11 (Krull [1928])**.** *Let K/k be a Galois extension and L/k a subextension. Then $\mathrm{Gal}(K/L)$ is a closed subgroup of $\mathrm{Gal}(K/k)$. Futhermore the maps*

$$L \mapsto \mathrm{Gal}(K/L)$$
$$H \mapsto K^H$$

give an inclusion-reversing bijection between subextensions $K/L/k$ and closed subgroups of $\mathrm{Gal}(K/k)$. A subextension is Galois if and only if the Galois group $\mathrm{Gal}(K/L)$ is normal in $\mathrm{Gal}(K/k)$. In this case we have

$$\mathrm{Gal}(L/k) \cong \mathrm{Gal}(K/k)/\mathrm{Gal}(K/L)$$

Remark 2.12. Notice that this theorem makes no assumption on K/k other than it being Galois; it can be a finite or an infinite extension. This theorem then encompasses the Main Theorem of Finite Galois Theory, because the finite Galois groups are given the discrete topology and thus all subgroups are closed.

2.2. **Étale algebras and Grothendieck's formulation of the main theorem of Galois theory.** Here we recall the definitions of étale k-algebras and how Grothendieck used them to reformulate the main theorem of infinite Galois theory due to Krull.

Throughout this chapter we fix a field, a seperable closure and an algebraic closure $k \subseteq k_s \subseteq \bar{k}$. The following reformulation of Galois theory is analogous (in a sense that will be made precise in the chapter about Galois categories) to the theory of covering spaces and the topological fundamental group. Morally we should think of the field k as a "space" and then fixing a seperable closure is akin to choosing a base point. We then consider seperable extensions $k \subseteq k' \subseteq k_s$ and those are analogous to connected coverings in topology.

Notation. We denote the absolute Galois group $\mathrm{Gal}(k_s/k)$ by $\mathrm{Gal}(k)$.

Definition 2.13. A finite dimensional k-algebra A is called (finite-) *étale* (over k) if it is isomorphic to a finite direct product of seperable extensions of k.

We denote by $\mathbf{F\acute{E}t}_k$ the full subcategory of $k - \mathrm{Alg}$ consisting of finite dimensional étale algebras.

An important classical way of characterizing finite seperable extensions of a field k is by looking at k-algebra homomorphisms from the extension to a fixed algebraic closure $\bar{k}$.

Lemma 2.14. *Let L/k be a finite extension and fix an algebraic closure $\bar{k}$ of k. Then there are at most $[L : k]$ k-algebra homomorphisms $L \to \bar{k}$. Equality is obtained if and only if L/k is seperable.*

Remark 2.15. We do not assume that $L \subset \bar{k}$.

We notice that if L/k is seperable and $\chi : L \to \bar{k}$ is a k-algebra homomorphism, then $\chi(L) \subset \bar{k}$ is a seperable extension of k and thus lies in k_s. With that in mind we see that applying

$$\mathrm{Hom}_k(-, k_s)$$

to finite seperable extensions yields a contravariant functor

$$\Phi : \mathcal{C} \to \mathbf{FinSet}$$

where $\mathcal{C}$ is the category of finite *seperable* extensions of k.

The images of Φ carry a natural left action of $\mathrm{Gal}(k)$ via

$$(g, \sigma) \mapsto g \circ \sigma$$

for $g \in \mathrm{Gal}(k)$ and $\sigma \in \mathrm{Hom}_k(L, k_s)$ for some seperable extension L/k.

Now we notice that if $\phi : L_1 \to L_2$ is a map between two seperable extension, then the image under Φ of ϕ is obtained by composing from the right

$$\Phi(\phi)(\sigma) = \sigma \circ \phi$$

for $\sigma \in \mathrm{Hom}_k(L_2, k_s)$. Therefore it is clear that each such $\Phi(\phi)$ is $\mathrm{Gal}(k)-$equivariant and we can consider Φ as a functor

$$\Phi : \mathcal{C} \to \mathrm{Gal}(k) - \mathbf{FinSet}$$

from the category of finite seperable extensions to the category of finite sets with an action of $\mathrm{Gal}(k)$.

As we stated before, $\mathrm{Gal}(k)$ is a profinite group. It is therefore natural to look at the discrete topology on these sets $\mathrm{Hom}_k(L, k_s)$ and ask if the action is continuous. The following lemma tells us that this $\mathrm{Gal}(k)$-action is indeed continuous and moreover that it is transitive.

Lemma 2.16. *This action of $\mathrm{Gal}(k)$ on $\mathrm{Hom}_k(L, k_s)$ is continuous and transitive, hence $\mathrm{Hom}_k(L, k_s)$ is isomorphic as a $\mathrm{Gal}(k)$-set to the left coset space of some open subgroup in $\mathrm{Gal}(k)$. When L/k is Galois, this coset space is in fact a quotient by an open normal subgroup.*

The lemma then tells us that we can view our functor Φ as

$$\Phi : \mathcal{C} \to \mathrm{Gal}(k) - \mathrm{Rep_c}$$

where $\mathrm{Gal}(k) - \mathrm{Rep_c}$ is the category of finite discrete sets with a continuous $\mathrm{Gal}(k)$-action.

This functor is indeed an anti-equivalence between $\mathcal{C}$ and the subcategory $\mathrm{Gal}(k) - \mathrm{Rep_c}^{tr}$ of $\mathrm{Gal}(k) - \mathrm{Rep_c}$ consisting of those sets in $\mathrm{Gal}(k) - \mathrm{Rep_c}$ that have a transitive action.

Theorem 2.17. *The contravariant functor Φ is an anti-equivalence from the category $\mathcal{C}$ of finite seperable extensions of k to the category* $\mathrm{Gal}(k) - \mathrm{Rep_c}^{tr}$*. Galois extensions give rise to* $\mathrm{Gal}(k)$*-sets that are isomorphic to some finite quotient of* $\mathrm{Gal}(k)$*.*

Now we wish to extend Φ to a functor

$$\Phi : \mathbf{F\acute{E}t}_k \to \mathrm{Gal}(k) - \mathrm{Rep_c}$$

in the obvious way. Namely let $A \in \mathbf{F\acute{E}t}_k$ be a finite dimensional étale k-algebra, then Φ is simply the functor sending A to $\mathrm{Hom}_k(A, k_s)$.

The important thing to notice here is that if A is a finite dimensional étale k-algebra, then it can by definition be written as

$$A = L_1 \times \ldots \times L_r$$

where $r \geq 1$ is some integer and all the L_i's are seperable extensions of k, and if $\phi \in \mathrm{Hom}_k(A, k_s)$ then it induces an injection $L_i \to k_s$ for some i. This is because if $\phi(L_i) \neq 0$ for some i then we have a k-algebra map $L_i \to k_s$ that of course is injective since L_i is a field. Furthermore if $j \neq i$ then we must have $\phi(L_j) = 0$. If not we would have two injections $L_i \to k_s$ and $L_j \to k_s$ and if we restrict ϕ to $L_i \times L_j$ we see that neither $(0, 1)$ nor $(1, 0)$ is mapped to 0 in k_s so we obtain zero-divisors in the field k_s, which is clearly absurd.

This tells us that $\mathrm{Hom}_k(A, k_s)$ splits as

$$\mathrm{Hom}_k(A, k_s) = \mathrm{Hom}_k(L_1, k_s) \amalg \ldots \amalg \mathrm{Hom}_k(L_r, k_s)$$

This composition $A = L_1 \times \ldots \times L_r$ thus corresponds to the splitting of a finite discrete set with a continuous $\mathrm{Gal}(k)$-action into the disjoint union of orbits. We then obtain Grothendieck's version of the fundamental theorem of Galois theory.

Theorem 2.18. *The functor Φ sending a finite étale k-algebra A to the set* $\mathrm{Hom}_k(A, k_s)$ *is an anti-equivalence from the category* $\boldsymbol{F\acute{E}t}_k$ *to the category* $\mathrm{Gal}(k) - \mathrm{Rep_c}$*. Seperable field extensions are sent to sets with a transitive* $\mathrm{Gal}(k)$ *action and Galois extensions are sent to sets that are isomorphic to a quotient of* $\mathrm{Gal}(k)$*.*

3. Fundamental groups and covering spaces in topology

In this section we let X denote a topological space. Recall that if $x \in X$ is a point, then the *fundamental group of X with basepoint x* is the group of all homotopy equivalences of loops in X based at x, and is denoted by $\pi_1(X, x)$. This formulation of the fundamental group does not give rise to an analogous construction in algebraic geometry. One of the main obstruction to this is the fact that the notion of a path is very problematic in algebraic geometry.

However we have 3.3 that tells us that in many cases the fundamental group is independent of the basepoint. Then we can reformulate the definition in terms of coverings and that reformulation lends itself much better to an algebro-geometric analogue.

First we recall a definition from topology.

Definition 3.1. A topological space X is said to be semilocally simply connected if every point $x \in X$ has a neighborhood U such that any loop in U based at x can be contracted in X to the point x.

Remark 3.2. Notice that the contraction does not have to stay within U so U does not have to be simply connected. Therefore this is a weaker property then being locally simply connected.

Theorem 3.3. *Let X be a connected, locally path-connected and semilocally simply connected topological space. Then $\pi_1(X, x)$ is independent of the choice of a base point x, up to non-canonical isomorphisms.*

In particular we have the following special case.

Corollary 3.4. *If X is a path-connected topological space, then $\pi_1(X, x)$ is independent of the choice of the base point x up to "inner isomorphisms".*

If X satisfies the above conditions, we denote the fundamental group simply by $\pi_1(X)$.

Definition 3.5. Let X and Y be topological spaces. A surjective continuous map $p : Y \to X$ is called a covering map if every point $x \in X$ has an open neighborhood U such that

$$p^{-1}(U) = \coprod_{\alpha} V_\alpha$$

where each V_α is homeomorphic to U. If $p : Y \to X$ is a covering map we also call Y a covering of X without explicitly mentioning the map p.

The fibre of p at each point is a discrete subspace of Y and for any two points in the same connected component in X these fibres are homeomorphic. In particular:

Proposition 3.6. *Let X be a connected topological space and $p : Y \to X$ a covering. Then there exists a discrete space F such that for each $x \in X$ the fibre $p^{-1}(x)$ is homeomorphic to F.*

From this proposition we see that in particular that if X is connected then the cardinality of the fibre is the same at each point. If this cardinality is finite then it is called the degree of the covering and we say that $p : Y \to X$ is a *finite covering*. If this cardinality is infinite we say that the covering is infinite.

We want to look at the *category* of coverings of a fixed space, so we need to have morphisms between them. If $p_1 : Y_1 \to X$ and $p_2 : Y_2 \to X$ are two coverings of X then a map of coverings is a continuous map $\phi : Y_1 \to Y_2$ such that the following diagram commutes.

$$\begin{tikzcd} Y_1 \arrow[r, "\phi"] \arrow[d, "p_1"'] & Y_2 \arrow[d, "p_2"] \\ X \arrow[r, equal] & X \end{tikzcd}$$

This allows us to make the following definition.

Definition 3.7. Let X be a connected topological space. Then the category $\mathrm{Cov}(X)$ is the category of all coverings of X and covering maps between them. The category $f\mathrm{Cov}(X)$ is the category of all finite coverings of X and covering maps between them.

If $p : Y \to X$ is a covering such that Y is a connected topological space, then we call the covering a connected covering.

Connected coverings will play an analogous role in this setting as seperable extensions do in Galois theory.

In Galois theory we study the continuous actions of groups on field extensions. Analogously, in the topological setting we want to study continuous actions of groups on spaces over a fixed base space X, or more precisely we want to study the continuous actions of groups on coverings $p : Y \to X$.

The following definition and lemma will tell us that given a certain type of group actions, we can associate a covering.

Definition 3.8. Let G be a topological group acting continuously on a topological space Y. The action of G is called *even* (or *properly discontinuous*) if for each point $y \in Y$ we can find an open neighbourhood U such that the open sets gU are pairwise disjoint for all $g \in G$.

Lemma 3.9. *Let G be a group acting evenly on a connected space Y. Then the projection*

$$p_G : Y \to G\backslash Y$$

gives Y the structure of a connected covering of $G\backslash Y$.

On the other hand, given a covering $p : Y \to X$ we want to associate with it a group. This is simply done by considering the *group of automorphisms of the covering*, i.e. the group of homeomorphisms $Y \to Y$ that are compatible with the covering projection p. We denote this group by $\mathrm{Aut}(Y/X)$.

If we now fix a base point $x \in X$, and consider a covering $p : Y \to X$, we get an induced group action of $\mathrm{Aut}(Y/X)$ on the fibre $p^{-1}(x)$.

We noted earlier how one obtains a connected covering from an even group action on a connected topological space. The following proposition tells us conversely how to get an even group action on a connected topological space from a connected covering.

Proposition 3.10. *Let $p : Y \to X$ be a connected covering. Then the action of* $\mathrm{Aut}(Y/X)$ *on Y is even.*

The following theorem then tells us that these operations are in some sense inverse of each other.

Proposition 3.11. *Let G be a group acting evenly on a connected topological space Y. Then the automorphism group* $\mathrm{Aut}(Y/(G\backslash Y))$ *of the connected covering $p_G : Y \to G\backslash Y$ is precisely G.*

We see a relationship between connected coverings and even group actions beginning to form. To explore this further we introduce a certain type of coverings, called Galois coverings.

In Galois theory we define Galois extensions of a field k to be an extension L/k for which the automorphism group $\mathrm{Gal}(L/k)$ fixes precisely k. Analogously in the topological setting we make the following definition.

Definition 3.12. Let $p : Y \to X$ be a connected cover with an automorphism group $\mathrm{Aut}(Y/X)$. This group acts on Y and the definition of a cover tells us that p factors through the quotient of Y by this action. That is, we have a map $\bar{p}$ such that the following diagram commutes.

$$\begin{tikzcd} Y \arrow[r, two heads] \arrow[rr, bend right, "p"'] & \mathrm{Aut}(Y/X)\backslash Y \arrow[r, "\bar{p}"] & X \end{tikzcd}$$

We say that $p : Y \to X$ is a *Galois covering* if this map $\bar{p}$ is a homeomorphism.

The following proposition tells us how this Galois property relates to the action of $\mathrm{Aut}(Y/X)$ on the fibres $p^{-1}(x)$.

Proposition 3.13. *A connected covering $p : Y \to X$ is Galois if and only if $\mathrm{Aut}(Y/X)$ acts transitively on each fibre $p^{-1}(x)$. In particular, if X is connected then the covering is Galois if and only if $\mathrm{Aut}(Y/X)$ acts transitively on any fibre.*

We are now in a position to state the topological analogue to the fundamental theorem of finite Galois theory.

Theorem 3.14. *Let $p : Y \to X$ be a Galois cover with automorphism group $G := \mathrm{Aut}(Y/X)$. For each subgroup $H \leq G$ the projection p induces a canonical map*

$$\bar{p}_H : H\backslash Y \to X$$

endowing $H\backslash Y$ the property of a covering of X.

Conversely, if $q : Z \to X$ is a covering that fits into a commutative diagram

$$\begin{tikzcd} Y \arrow[r, "f"] \arrow[dr, "p"'] & Z \arrow[d, "q"] \\ & X \end{tikzcd}$$

Then $f : Y \to Z$ is a Galois covering and

$$Z \cong H\backslash Y$$

where $H := \mathrm{Aut}(Y/Z) \leq G$. These maps

$$H \mapsto H\backslash Y$$
$$Z \mapsto \mathrm{Aut}(Y/Z)$$

give a bijection between subgroups of G and intermediate coverings $q : Z \to X$, i.e. coverings that fit into the commutative diagram above. Furthermore, the covering $q : Z \to X$ is Galois if and only if H is a normal subgroup of G. In that case we have

$$\mathrm{Aut}(Z/X) \cong G/H$$

In Grothendieck's formulation of the fundamental theorem of Galois theory, we had an equivalence between the category $\mathbf{F\acute{E}t}_k$ of finite étale algebras over k and the category $\mathrm{Gal}(k) - \mathrm{Rep_c}$ of finite sets with a continuous action of the absolute Galois group of k. To find an analogue of this theorem in the topological setting, we need a topological analogue of the absolute Galois group, i.e. we need some "absolute automorphism group of covers of X".

An obvious candidate for this "absolute automorphism group" is the fundamental group of X. It turns out not to be the right choice, but a modification of it is.

Let us recall the Homotopy Lifting Lemma.

Lemma 3.15. *Let X be a topological space and $x \in X$ be a fixed basepoint. Let $p : Y \to X$ be a covering of X and $y \in p^{-1}(x) \subset Y$ be a point in the fibre over x. Then:*

(1) If $f : [0,1] \to X$ is a path beginning at $f(0) = x$, then there is a unique path $\tilde{f} : [0,1] \to Y$ that is a lift of f beginning at y. That is we have the $\tilde{f}(0) = y$ and the following commutative diagram

$$\begin{tikzcd} {[0,1]} \arrow[r, dashed, "\tilde{f}"] \arrow[dr, "f"'] & Y \arrow[d, "p"] \\ & X \end{tikzcd}$$

(2) Given two homotopic paths $f, g : [0,1] \to X$ the unique liftings $\tilde{f}, \tilde{g}$ of f and g respectively beginning at y, are homotopic.

The first part of the Homotopy Lifting Lemma tells us that if we take a loop α based at $x \in X$ then the unique path $\tilde{\alpha}$ in Y beginning at some $y \in p^{-1}$ must end in a point $z \in Y$ such that

$$p(z) = p \circ \tilde{\alpha}(1) = \alpha(1) = x$$

i.e. $z \in p^{-1}(x)$.

The second part of the lemma then tells us that if $[\alpha]$ is an element in the fundamental group $\pi_1(X, x)$ then any choice of a representative $\alpha : [0,1] \to X$ gives the same point $z \in p^{-1}(x)$.

We thus have an action of the fundamental group $\pi_1(X, x)$ on the fibre $p^{-1}(x)$. This is called the *monodromy* action.

Thus we can look at the fundamental group as a group of *deck transformations* of the covering space $p : Y \to X$. What we mean by this is that there is an open neighbourhood U of x that trivializes p, i.e. such that

$$p^{-1}(U) = \coprod_{i \in I} V_i$$

where each V_i is homeomorphic to U. This Monodromy action then "shuffles" these V_i around.

Now, just like $\mathrm{Gal}(k)$ was the automorphism group of a unique (up to isomorphisms) extension, namely the seperable closure k_s, we want to find a covering for which the "absolute automorphism group" we are searching for is the automorphism group.

To this end, we recall the definition and some properties of *universal coverings*.

Definition 3.16. Let X be a topological space with a fixed base point x. A pointed space $(\tilde{X}_x, \tilde{x})$ is called a *universal covering of* X *with respect to* x if there exists a map $\pi : \tilde{X} \to X$ making $\tilde{X}$ a covering of X and such that $\tilde{x}$ lies in the fibre of π over x, and such that for any such pointed covering (Y, y) of X with covering map p there is a map

$$\tilde{p} : \tilde{X} \to Y$$

endowing $\tilde{X}$ with the structure of a covering of Y and such that the following diagram commutes

$$\begin{tikzcd} \tilde{X} \arrow[r, dashed, "\pi"'] \arrow[dr, "\tilde{p}"] & Y \arrow[d, "p"] \\ & X \end{tikzcd}$$

The important property of the universal covering (as pertains to us) is that for connected and locally simply connected spaces it represents the fibre functor

$$\mathrm{Fib}_x : \mathrm{Cov}(X) \to \mathbf{Set}$$

that takes a covering $p : Y \to X$ to the fibre of p over x.

Theorem 3.17. *The groups* $\pi_1(X, x)$ *and* $\mathrm{Aut}(\tilde{X}_x/X)^{op}$ *are naturally isomorphic.*

By construction the group $\mathrm{Aut}(\tilde{X}_x/X)$ has a natural left action on $\tilde{X}_x$. This induces a right action of $\mathrm{Aut}(\tilde{X}_x/X)$ or equivalently a left action of $\mathrm{Aut}(\tilde{X}_x/X)^{op}$ on $\mathrm{Hom}_X(\tilde{X}_x, Y) \cong \mathrm{Fib}_x(Y)$.

Now we can state a theorem that shows how in the topological setting, $\tilde{X}_x$ is analogous to a seperable closure in the Galois setting.

Theorem 3.18. *Let* X *be a topological space and* $x \in X$ *a base point. Then the cover* $\pi : \tilde{X}_x \to X$ *is a connected Galois cover. Furthermore, for an arbitrary cover* $p : Y \to X$ *the left action of* $\mathrm{Aut}(\tilde{X}_x/X)^{op}$ *on* $\mathrm{Fib}_x(Y)$ *is exactly the monodromy action of* $\pi_1(X, x)$*.*

Theorem 3.19. *Let* X *be a connected, locally path-connected and semilocally simply connected topological space. Then the fibre functor induces an equivalence of categories*

$$f\mathrm{Cov}(X) \cong \widehat{\pi_1(X)} - \mathrm{Rep}_{\mathrm{c}}$$

where $\widehat{\pi_1(X)} - \mathrm{Rep}_{\mathrm{c}}$ *is the category of all finite groups with the discrete topology endowed with a continuous action of* $\widehat{\pi_1(X)}$*.*

This profinite completion $\widehat{\pi_1(X)}$ is the right analogue for the étale fundamental group in algebraic geometry.

3.1. **Locally constant sheaves and coverings.** In the last section of this chapter we want to reformulate 3.19 in terms of locally constant sheaves on X. When we generalize sheaves on topological spaces to sheaves on sites, this will be the theorem we shall find an analogue for.

Definition 3.20. A sheaf $\mathcal{F}$ on X is said to be locally constant if for each $x \in X$ there exists a neighborhood U of x such that the restricted sheaf $\mathcal{F}_U$ is (isomorphic to) a constant sheaf.

Now if we are given a covering $p : Y \to X$ and an open set $U \subset X$ we define as usual a section of p over U to be a map

$$s : U \to Y$$

such that

$$p \circ s = id_U$$

If $V \subset U$ and $s : U \to Y$ is a section of p over U then the restriction $s|_V : V \to Y$ is obviously a section of p over V.

This allows us to define a presheaf og sets $\mathcal{F}_Y$ on X associated with the given covering, in the following manner

$$\mathcal{F}_Y(U) = \{\text{sections of } p \text{ over } U\}$$

Proposition 3.21. *The presheaf $\mathcal{F}_Y$ defined above is a locally constant sheaf. It is a constant sheaf if and only if $p : Y \to X$ is a trivial covering.*

Thus we have an assignment

$$Y \mapsto \mathcal{F}_Y$$

sending a covering to a locally constant sheaf is functorial and moreover

Proposition 3.22. *The functor defined above induces an equivalence*

$$\mathrm{Cov}(X) \cong \mathrm{Loc}(X)$$

between the category of coverings of X and the category of locally constant sheaves on X.

Now we have the following equivalences:

$$\mathrm{Loc}(X) \cong \mathrm{Cov}(X) \cong \pi_1(X, x) - \mathbf{Set}$$

the composed functor inducing the equivalence

$$\mathrm{Loc}(X) \cong \pi_1(X, x) - \mathbf{Set}$$

is simply just the functor of taking stalks at x.

Restricting this functor to finite locally constant sheaves finally gives us

Theorem 3.23. *Let X be a connected, locally path-connected and semilocally simply connected topological space. Then we have an equivalence of categories.*

$$f\mathrm{Loc}(X) \cong \widehat{\pi_1(X)} - \mathrm{Rep_c}$$

4. Galois categories

4.1. **Galois categories and fundamental groups.** In this section we follow the exposition in Cadoret [2013] closely and recommend it for further detail and clarity.

Definition 4.1. Let X be an object of a category $\mathcal{C}$ and let $G \in \mathrm{Aut}_{\mathcal{C}}(X)$ be a finite group of automorphisms of X. The *quotient of X by G* is defined (if it exists) as a pair $(X/G, \rho)$ where X/G is an object in $\mathcal{C}$ and $\rho : X \to X/G$ is a morphism such that $\rho\sigma = \rho$ for any $\sigma \in G$, satisfying the universal property of making the following diagram commutative

$$\begin{tikzcd}
X \arrow[rr, "\sigma"] \arrow[dr, "\rho"] \arrow[ddr, "f"'] & & X \arrow[dl, "\rho"'] \arrow[ddl, "f"] \\
& X/G \arrow[d, dashed, "\exists! u"] & \\
& Z &
\end{tikzcd}$$

Where Z is an object of $\mathcal{C}$ and $f : X \to Z$ is a morphism such that $f\sigma = f$ for all $\sigma \in G$.

Morally we should think of the universal property as saying that "any morphism that commutes with the G-action, factors through the orbit-space/orbit-set".

Example 4.2. Consider the category **FinSet**. An object $X \in \mathbf{FinSet}$ is then a finite set and the group $\mathrm{Aut}_{\mathbf{FinSet}}(X)$ is isomorphic to the symmetric group S_n, where $n = |X|$. Now if G is a finite subgroup of automorphisms of X, then we let X/G be the finite sets of orbits of the action of G on X and $X \xrightarrow{\rho} X/G$ is the map that sends $x \in X$ to its orbit. Now clearly $\rho \circ \sigma(x) = \rho(x)$ for all x. If Z is some finite set and $X \xrightarrow{f} Z$ is a map that commutes with the G-action, then we can uniquely define a map $X/G \xrightarrow{u} Z$ by sending the orbit of an element x to $f(x)$. This is well defined and by construction f factors through X/G.

Definition 4.3. A morphism $u : X \to Y$ in a category $\mathcal{C}$ is called a *strict epimorphism* if the pullback

$$\begin{tikzcd} X \times_Y X \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & X \arrow[d, "u"] \\ X \arrow[r, "u"'] & Y \end{tikzcd}$$

exists and for any object $Z \in \mathcal{C}$, the map

$$\circ u : \mathrm{Hom}_{\mathcal{C}}(Y, Z) \to \mathrm{Hom}_{\mathcal{C}}(X, Z)$$

is injective and induces a bijection onto the set of all morphisms $f : X \to Z$ such that $f \circ \pi_1 = f \circ \pi_2$.

Definition 4.4 (Galois Category)**.** A *Galois category* is a category $\mathcal{C}$ along that admits a covariant functor

$$F : \mathcal{C} \to \mathbf{FinSet}$$

called a *fibre functor for* $\mathcal{C}$, such that the following axioms are satisfied:

(1) $\mathcal{C}$ has a final object $e_{\mathcal{C}}$ and finite fibre products exist in $\mathcal{C}$.
(2) Finite coproducts exist in $\mathcal{C}$ and categorical quotients by finite groups of automorphisms exist in $\mathcal{C}$.
(3) Any morphism $u : Y \to X$ in $\mathcal{C}$ factors as

$$Y \xrightarrow{u'} X' \xrightarrow{u''} X$$

where u' is a strict epimorphism and u'' is a monomorphism which is an isomorphism onto a direct summand of X.
(4) F sends final objects to final objects and commutes with fibre products.
(5) F commutes with finite coproducts and categorical quotients by finite groups of automorphisms and sends strict epimorphisms to strict epimorphisms.
(6) Let $u : Y \to X$ be a morphism in $\mathcal{C}$, then $F(u)$ is an isomorphism if and only if u is an isomorphism.

Remark 4.5. Directly from the axioms (more specifically axioms (3), (4) and (6)) we obtain that if $\mathcal{C}$ is a Galois category with a fibre functor $F : \mathcal{C} \to \mathbf{FinSet}$ then a morphism u in $\mathcal{C}$ is a strict epimorphism or resp. a monomorphisms if and only if $F(u)$ is a strict epimorphism or resp. a monomorphism. And for a given object $X \in \mathcal{C}$:

(a) $F(X) = \emptyset$ if and only if X is an initial object in $\mathcal{C}$.
(b) $F(X) \cong \{\star\}$ if and only if X is a final object in $\mathcal{C}$.

(here $\{\star\}$ denotes the (unique up to a unique isomorphism) one element set in **FinSet**)

Definition 4.6. Given a Galois category $\mathcal{C}$ with a fibre functor F, we call the automorphism group $\mathrm{Aut}(F)$ (in $Fun(\mathcal{C})$, the category of functors $\mathcal{C} \to \mathbf{Set}$) *the fundamental group of* $\mathcal{C}$ *at* F.

Another thing to notice is that any Galois category is Artinian. Recall that a category $\mathcal{C}$ is said to be Artinian if for every object X every descending chain of subobjects

$$\ldots \subseteq X_{i+1} \subseteq X_i \subseteq \ldots \subseteq X_1 \subseteq X$$

stabilizes.

As with any definition we want an example to illuminate it. It turns out that the following example is fundamental to the theory of Galois categories, as we shall see when we state and prove the main theorems on Galois categories.

Example 4.7. Let π be a profinite group. The category

$$\mathcal{C}(\pi) := \pi - \mathrm{Rep_c}$$

is a Galois category with a fibre functor

$$For : \mathcal{C}(\pi) \to \mathbf{FinSet}$$

the forgetful functor to the category of finite sets.

We prove this by going through the axioms.

(1) Consider two morphisms, $f : Y \to X$ and $g : Z \to X$ in $\mathcal{C}(\pi)$. Consider the set

$$Y \times_X Z := \{(y, z) \in Y \times Z \mid f(y) = g(z)\}$$

Now $Y \times Z$ has a natural continuous action of π, namely the diagonal action

$$\gamma(y, z) = (\gamma y, \gamma z)$$

and it's clear that the subset $Y \times_X Z$ is actually a sub-π-set. It suffices to show that it is closed under the action of π. So let $(y, z) \in Y \times_X Z$ and $\gamma \in \pi$ then, because f and g are intertwiners, we have

$$f(\gamma y) = \gamma f(y) = \gamma g(z) = g(\gamma z)$$

It's easy to check that this satisfies the universal property of the fibre product. A singleton with the trivial action of π is a final object for $\mathcal{C}(\pi)$

(2) The coproduct in $\mathcal{C}(\pi)$ is the disjoint union. Finite disjoint unions clearly exist in $\mathcal{C}(\pi)$.

Now let $X \in \mathcal{C}(\pi)$ and consider a finite group G of automorphisms of X. Then G consists of bijections $\gamma : X \to X$ that commute with the action of π. Now let X/G be the set consisting of the orbits of the action of G on X, and $\rho : X \to X/G$ be the projection onto the orbits. We want to show that X/G has a natural continuous action of π that commutes with ρ. Let us denote the equivalence class of x in X/G by $[x]$ and define an action of π on X/G by

$$\gamma[x] = [\gamma x]$$

For all $\gamma \in \pi$. Now assume $[x] = [y]$. Then there exists a $g \in G$ such that $y = g(x)$. Then

$$\begin{aligned}\gamma[y] = [\gamma y] = [\gamma g(x)] &= [g(\gamma x)] \\ &= [\gamma x] \\ &=: \gamma[x]\end{aligned}$$

Where the second to last equation comes simply from the definition of equivalence classes. We need to show that the map

$$\phi_G : \pi \times X/G \to X/G$$

is continuous. This follows from the fact that the quotient map $\rho : X \to X/G$ is open and f fits into the commutative diagram

$$\begin{tikzcd} \pi \times X \arrow[r, "(id,\rho)"] \arrow[d, "\phi"'] & \pi \times X/G \arrow[d, "\phi_G"] \\ X \arrow[r, "\rho"'] & X/G \end{tikzcd}$$

where ϕ, ρ and (id, ρ) are all continuous.

(3) Let $f : Y \to X$ be a map in $\mathcal{C}(\pi)$. We can factor f as a map of finite sets as

$$Y \xrightarrow{f} f(Y) \xhookrightarrow{i} X$$

where $i : f(Y) \to X$ is the inclusion. Now π acts continuously on $f(Y)$ and commutes with f by assumption, so

$$f : Y \to f(Y)$$

is an epimorphism in $\pi - \mathrm{Rep_c}$. Clearly $i : f(Y) \to X$ commutes with the action of π and so i is an injective map, i.e. a monomorphism, in $\pi - \mathrm{Rep_c}$. What remains to be seen is that

$$f : Y \to f(Y)$$

is a strict epimorphism.

The fibre product

$$\begin{array}{ccc} Y \times_{f(Y)} Y & \xrightarrow{p_2} & Y \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle f} \\ Y & \xrightarrow{f} & f(Y) \end{array}$$

exists in $\pi - \mathrm{Rep_c}$, it can be realized as

$$Y \times_{f(Y)} Y = \coprod_{y \in Y} \{y\} \times f^{-1}(y)$$

with the obvious action of π induced from the action on $Y \times Y$. Now if Z is any discrete finite set with a continuous π-action, then we look at the map on Hom's given by

$$\circ f : \mathrm{Hom}(f(Y), Z) \to \mathrm{Hom}(Y, Z)$$

This is clearly injective; if $g_1, g_2 \in \mathrm{Hom}(f(Y), Z)$ are maps such that

$$g_1 \circ f = g_2 \circ f$$

then by definition

$$g_1(f(y)) = g_2(f(y))$$

for all $y \in Y$. That is to say, g_1 and g_2 agree on all elements in $f(Y)$ and are therefore the same.

The fact that this $\circ f$ induces a bijection onto the set of $\psi \in \mathrm{Hom}(Y, Z)$ such that

$$\psi \circ p_1 = \psi \circ p_2$$

is clear from construction.

(4) The final objects in $\pi - \mathrm{Rep_c}$ are the one-point sets with the trivial action of π. The forgetful functor sends this set to a one-point set, i.e. a final object in **FinSet**.

It is clear from the construction of the fibre product in $\pi - \mathrm{Rep_c}$ that it is simply the fibre product of the underlying sets with the extra structure of a continuous action of π. Therefore the forgetful functor commutes with taking fibre products.

(5) Again, it is clear that coproducts and quotients by finite groups of automorphisms in $\pi - \mathrm{Rep_c}$ are by construction the coproducts and quotients by finite group of automorphisms in **FinSet**

with the extra structure of a continuous action of π and so clearly commute with the forgetful functor.

The strict epimorphisms of **FinSet** are simply the epimorphisms, so any strict epimorphism in $\pi - \mathrm{Rep_c}$ is sent by the forgetful functor to a strict epimorphism.

(6) Isomorphisms in $\pi - \mathrm{Rep_c}$ are the bijections of the underlying set that commute with the action of π. Therefore a π-map

$$f : Y \to X$$

is an isomorphism if and only if it is bijective, i.e. f is an isomorphims if and only if $For(f)$ is an isomorphism.

Notation. From now on we fix a Galois category $\mathcal{C}$ and a fibre functor $F : \mathcal{C} \to \mathbf{FinSet}$, unless we explicitly state otherwise.

Our aim is to state and prove *the main theorem on Galois categories* that roughly says that up to equivalences the Galois categories $\mathcal{C}(\pi)$ associated with profinite groups π along with the forgetful functor, are the only Galois categories and fibre functors there are, and up to non-canonical isomorphisms, the choice of a fibre functor gives the same fundamental group and thus an equivalence of a Galois category $\mathcal{C}$ with a category of the form $\mathcal{C}(\pi)$ does not depend on the choice of a fibre functor.

To this end we introduce some notions that are analogous to notions from Galois theory.

Definition 4.8. Given two objects $X, Y \in \mathcal{C}$ we say that X *dominates* Y if there exists at least one morphism

$$X \to Y$$

and we write this as $X \geq Y$.

Definition 4.9. We say that an object $X \in \mathcal{C}$ is connected if it can not be written as a coproduct $X = X_1 \coprod X_2$ of two objects $X_1, X_2 \in \mathcal{C}$ neither of which is an initial object.

Like in topology we get a decomposition into connected components. Unlike the general case in topology, this decomposition here is always finite because $\mathcal{C}$ is Artinian (this is similar to the condition of Noetherianness on a topological space).

Remark 4.10. From now on we'll denote the unique initial object in a category $\mathcal{D}$ by $I_{\mathcal{D}}$, if it exists.

Proposition 4.11. *An object $X_0 \in \mathcal{C}$ is connected if and only if for any non-initial object $X \in \mathcal{C}$ any monomorphism $X \to X_0$ is an isomorphism. In particular, any non-initial object $X \in \mathcal{C}$ can be decomposed as*

$$X = \coprod_{i=1}^{n} X_i$$

where X_i is a connected, non-initial object for any i. This decomposition is unique (up to permutation of the connected components).

Proof. We start by assuming that we have an object X_0 given, such that for any $X \in \mathcal{C}, X \neq I_{\mathcal{C}}$ any monomorphism $i : X \to X_0$ is an isomorphism. Now assume we decompose X_0 as a coproduct of two objects

$$X_0 = X_0^{'} \amalg X_0^{''}$$

where we assume without loss of generality that $X_0^{'} \neq I_{\mathcal{C}}$. We have a canonical morphism

$$\iota_{X_0^{'}} : X_0^{'} \to X_0$$

and by 4.5 we see that it is monic. Hence by assumption it is an isomorphism and we have $F(X_0^{''}) = \emptyset$ and thus by the same remark $X_0^{''} = I_{\mathcal{C}}$ and X_0 is connected.

Now assume that we are given a connected object $X_0 \neq I_{\mathcal{C}}$ of $\mathcal{C}$. The third axiom tells us that any monomorphism $i : X \to X_0$ factors as

$$\begin{tikzcd} X \arrow[r, two heads, "i^{'}"] \arrow[rr, bend right, "i"'] & X_0^{'} \arrow[r, "i^{''}"] & X_0 = X_0^{'} \amalg X_0^{''} \end{tikzcd}$$

where $i^{'}$ is a strict epimorphism and $i^{''}$ is a monomorphism inducing an isomorphism on the factor $X_0^{'}$. If $X_0^{'} = I_{\mathcal{C}}$ then $F(X) = \emptyset$ and hence $X = I_{\mathcal{C}}$, contradicting our assumptions on X. Thus the connectedness of X_0 gives that $X_0^{''} = I_{\mathcal{C}}$ and so $i^{''}$ is an isomorphism. Then $i : X \to X_0$ is a monomorphism and a strict epimorphism, and thus an isomorphism.

To show the decomposition part of the theorem we assume we have an object $X \in \mathcal{C}$, $X \neq I_{\mathcal{C}}$ and notice that since $\mathcal{C}$ is artinian there exists some $X_1 \in \mathcal{C}$ such that $X_1 \neq I_{\mathcal{C}}$, X_1 is connected and we have a monomorphism

$$i_1 : X_1 \hookrightarrow X$$

If i_q is an isomorphism, then X is connected and we are done. Otherwise, the third axiom tells us that i_1 splits as

$$\begin{tikzcd} X_1 \arrow[r, two heads, "i_1'"] \arrow[rr, bend right, "i_1"'] & X' \arrow[r, "i_1''"] & X = X' \amalg X'' \end{tikzcd}$$

where i_1' is a strict epimorphism and i_1'' is a monomorphism that induces an isomorphism onto the factor X'. Now since i_1 and i_1'' are both monomorphisms, then so is i_1'. But i_1' is also a strict epimorphism and thus it's an isomorphism. Now we have decomposed

$$X = X_1 \amalg X''$$

and we repeat the process for X''. The artinian property assures us that this process will terminate in a finite number of steps and thus we can write

$$X = X_1 \amalg \ldots \amalg X_n$$

where $n \geq 1$ and all the X_i's are connected and non-initial objects.

Now the only thing that remains is to show that this decomposition is unique. So we assume we have another such decomposition

$$X = Y_1 \amalg \ldots \amalg Y_m$$

where $m \geq 1$ and all the Y_j's are connected and not initial objects. For any $1 \leq i \leq n$ there exists some $1 \leq \sigma(i) \leq m$ such that $F(X_i) \cap F(Y_{\sigma(i)}) \neq \emptyset$. Now consider that we have canonical monomorphisms

$$\begin{aligned} i_{X_i} &: X_i \hookrightarrow X \\ i_{Y_{(\sigma(i))}} &: Y_{\sigma(i)} \hookrightarrow X \end{aligned}$$

and we can thus form the fibre product in $\mathcal{C}$ and consider the commutative diagram

$$\begin{tikzcd} X_i \times_X Y_{\sigma(i)} \arrow[r, "p_2"] \arrow[d, "p_1"'] & Y_{\sigma(i)} \arrow[d, hook, "i_{Y_{\sigma(i)}}"] \\ X_i \arrow[r, hook, "i_{X_i}"'] & X \end{tikzcd}$$

The fibre functor preserves fibre products so we consider the commutative diagram in **FinSet**

$$\begin{tikzcd} F(X_i) \times_{F(X)} F(Y_{\sigma(i)}) \arrow[r, "F(p_2)"] \arrow[d, "F(p_1)"'] & F(Y_{\sigma(i)}) \arrow[d, hook, "F(i_{Y_{\sigma(i)}})"] \\ F(X_i) \arrow[r, hook, "F(i_{X_i})"'] & F(X) \end{tikzcd}$$

In **FinSet** the fibre product is the equalizer of p_1 and p_2, i.e. the subset of the cartesian product $F(X_i) \times F(Y_{\sigma(i)})$ consisting of pairs (x, y) such that

$$F(i_{X_i})(x) = F(i_{Y_{\sigma(i)}})(y)$$

and the maps $F(p_1)$ and $F(p_2)$ are the restrictions of the projection maps π_1 and π_2 respectively. Therefore if we have (x_1, y_1) and (x_2, y_2) in $F(X_i) \times_{F(X)} F(Y_{\sigma(i)})$ such that $F(p_2)(x_1, y_1) = F(p_2)(x_2, y_2)$ then $y_1 = y_2$ so we may say $y_1 = y_2 =: y$. Now the square is commutative so

$$F(i_{X_i}) \circ F(p_1)(x_1, y) = F(i_{X_i}) \circ F(p_1)(x_2, y) = F(i_{Y_{\sigma(i)}})(y)$$

i.e.

$$F(i_{X_i})(x_1) = F(i_{X_i})(x_2).$$

The fibre functor also preserves monomorphisms so the map $F(i_{X_i})$ is injective and so $x_1 = x_2$. This shows that $F(p_2)$ is injective and so p_2 is a monomorphism. Since by construction we are taking the fibre product of two inclusions (once we've applied the fibre functor) in **FinSet**, we have simply that

$$F(X_i) \times_{F(X)} F(Y_{\sigma(i)}) = F(X_i) \cap F(Y_{\sigma(i)})$$

But by choice of $Y_{\sigma(i)}$ this intersection is nonempty. Therefore we have

$$X_i \times_X Y_{\sigma(i)} \neq I_{\mathcal{C}}$$

So we now have a monomorphism p_2 from a noninitial object to a connected object, hence an isomorphism. Similarly p_1 is an isomorphism and thus

$$X_i \cong Y_{\sigma(i)}$$

□

Now we introduce *the pointed category associated with the Galois category $\mathcal{C}$ and fibre functor F*. The idea behind it is to replace a category $\mathcal{C}$ by a category $\mathcal{C}^{pt}$ with more objects but fewer morphisms between the objects. This is a familiar process, for example from topology where we can look at coverings of a space X and maps between them or choose a base point $x \in X$ and for the pair (X, x) we consider coverings to be pairs (Y, y) where $p : Y \to X$ is a covering and y lies in the fibre of p over x. Now maps between coverings must be covering maps in the classical sense that send the chosen point of the source to the chosen point of the target.

Definition 4.12. Associated to the Galois category $\mathcal{C}$ and fibre functor F is a category $\mathcal{C}^{pt}$ whose objects are pairs (X, ζ) where $X \in \mathcal{C}$ and $\zeta \in F(X)$, and whose morphisms from (X_1, ζ_1) to (X_2, ζ_2) are precisely those $\mathcal{C}$-morphisms $u : X_1 \to X_2$ for which

$$F(u)(\zeta_1) = \zeta_2$$

There is a natural forgetful functor

$$For : \mathcal{C}^{pt} \to \mathcal{C}$$

and a 1-to-1 correspondence between sections of $For : Ob(\mathcal{C}^{pt}) \to Ob(\mathcal{C})$ and families

$$\underline{\zeta} = (\zeta_X)_{X \in Ob(\mathcal{C})} \in \prod_{X \in Ob(\mathcal{C})} F(X)$$

Now we consider morphisms to and from connected objects.

Proposition 4.13. *Let X_0 be a connected object and X any object in $\mathcal{C}$. Then:*

(1) ***(Rigidity)*** *For any $\zeta_0 \in F(X_0)$ and $\zeta \in F(X)$ there is at most one morphism from (X_0, ζ_0) to (X, ζ) in the pointed category $\mathcal{C}^{pt}$.*

(2) ***(Domination by connected objects)*** *For any finite family $\{(X_i, \zeta_i)\}_{i=1}^n$ of objects in $\mathcal{C}^{pt}$ there exists an object $(X_0, \zeta_0) \in \mathcal{C}^{pt}$ with $X_0 \in \mathcal{C}$ connected, such that*

$$(X_0, \zeta_0) \geq (X_i, \zeta_i)$$

for all $1 \leq i \leq n$. In particular, for any $X \in \mathcal{C}$ there exists an object (X_0, ζ_0) in the pointed category $\mathcal{C}^{pt}$ with X_0 connected, such that the evaluation map

$$\begin{aligned} ev_{\zeta_0} : \mathrm{Hom}_{\mathcal{C}}(X_0, X) &\to F(X) \\ \{u : X_0 \to X\} &\mapsto F(u)(\zeta_0) \end{aligned}$$

is a bijection.

(3) (a) *If $X_0 \in \mathcal{C}$ is a connected object, then any morphism $u : X \to X_0$ is a strict epimorphism.*

(b) *If $u : X_0 \to X$ is a strict epimorphism, with X_0 connected, then X is also connected.*

(c) Any endomorphism $u : X_0 \to X_0$ of a connected object is automatically an automorphism.

Proof. (1) Consider two morphisms in $\mathcal{C}^{pt}$

$$u_i : (X_0, \zeta_o) \to (X, \zeta)$$

where $i = 1, 2$. By the first axiom the equalizer

$$Eq(u_1, u_2) \xrightarrow{\iota} X$$

exists in $\mathcal{C}$ and the fourth axiom tells us that the fibre functor preserves this equalizer, namely

$$F(Eq(u_1, u_2)) \xrightarrow{F(\iota)} F(X)$$

is the equalizer of the maps $F(u_i) : F(X_0) \to F(X)$ in **FinSet**. Now by assumption we have $\zeta_0 \in Eq(F(u_1), F(u_2)) = F(Eq(u_1, u_2))$ so in particular $F(Eq(u_1, u_2)) \neq \emptyset$. Therefore, since F preserves initial objects, we have $Eq(u_1, u_2) \neq I_{\mathcal{C}}$. An equalizer is always a monomorphism so it follows from 4.11 that it is an isomorphism. Hence $u_1 = u_2$.

(2) Say we are given such a family $\{(X_i, \zeta_i)\}_{i=1}^n$ and consider the object

$$X := X_1 \times \ldots \times X_n$$

that exists in $\mathcal{C}$ according to the first axiom (consider the cartesian product as the fibred product over a final object). Also we set

$$\begin{aligned} \zeta := (\zeta_1, \ldots, \zeta_n) &\in F(X_1) \times \ldots \times F(X_n) \\ &= F(X_1 \times \ldots \times X_n) \\ &= F(X) \end{aligned}$$

Now the i-th projection $\pi_i : X \to X_i$ induces a morphism $(X, \zeta) \to (X_i, \zeta_i)$ in $\mathcal{C}^{pt}$ and so it suffices to find an object $(X_0, \zeta_o) \in \mathcal{C}^{pt}$ such that X_0 is a connected object in $\mathcal{C}$ and

$$(X_0, \zeta_0) \geq (X, \zeta)$$

If X is connected then we are done. So we assume it is not and write X as a coproduct of its connected components

$$X = \coprod_{j=1}^{m} X_j'$$

and as before we let $i_{X_j'} : X_j' \hookrightarrow X$ denote the canonical monomorphism. The second axiom tells us that the fibre functor preserves finite coproducts and therefore

$$F(X) = \coprod_{j=1}^{m} F(X_j')$$

where the coproduct in the second equation is the coproduct in **FinSet** i.e. the disjoint union. Therefore there exists a unique j such that $\zeta \in F(X_j')$ and thus $X_0 := X_j'$ and $\zeta_0 := \zeta$ gives us the desired element in $\mathcal{C}^{pt}$ that dominates (X, ζ).

(3) (a) Any morphism $u : X \to X_0$ factors, according to the third axiom, as

$$\begin{tikzcd} X \arrow[r, two heads, "u^{'}"] \arrow[rr, bend right, "u"'] & X_0^{'} \arrow[r, "u^{''}"] & X_0 = X_0^{'} \amalg X_0^{''} \end{tikzcd}$$

where $u^{'}$ is a strict epimorphism and $u^{''}$ is a monomorphism that induces an isomorphism onto the factor $X_0^{'}$ of X_0. Now by assumption $X \neq I_{\mathcal{C}}$ and therefore $X_0^{'} \neq I_{\mathcal{C}}$. But since X_0 is connected we must have $X_0^{''} = I_{\mathcal{C}}$ and therefore

$$u^{''} : X_0^{'} \xrightarrow{\cong} X_0$$

is an isomorphism and so $u : X \to X_0$ is a strict epimorphsm.

(b) Now we let X_0 be a connected object and $u : X_0 \to X$ be a strict epimorphism in $\mathcal{C}$. If $X_0 = I_{\mathcal{C}}$ then we have nothing to show. We therefore assume $X_0 \neq I_{\mathcal{C}}$. Let us write $X = X^{'} \amalg X^{''}$ where we assume $X^{'} \neq I_{\mathcal{C}}$, and as before we denote the canonical monomorphism $X^{'} \hookrightarrow X$ by $i_{X'}$. We also fix $\zeta^{'} \in F(X^{'})$ and $\zeta_0 \in F(X_0)$ such that $F(u)(\zeta_0) = \zeta^{'}$. By part 2 of this proposition there exists some $(X_0^{'}, \zeta_0^{'}) \in \mathcal{C}^{pt}$ with $X_0^{'} \in \mathcal{C}$ connected, that dominates (X_0, ζ_0) and $(X^{'}, \zeta^{'})$. I.e. there are morphisms

$$\phi_0 : (X_0^{'}, \zeta_0^{'}) \to (X_0, \zeta_0)$$
$$\phi : (X_0^{'}, \zeta_0^{'}) \to (X^{'}, \zeta^{'})$$

Now from 3a above we see that ϕ_0 is a strict epimorphism so the composition $u \circ \phi_0$ is a strict epimorphism as well. From 1 we have

$$u \circ \phi_0 = i_{X'} \circ \phi$$

and so $i_{X'} \circ \phi$ is in particular also a strict epimorphism. Therefore $F(X) = F(X^{'})$ which implies that $F(X^{''}) = \emptyset$. Fibre functors preserving initial objects then tells us that $X^{''} = I_{\mathcal{C}}$ and X is connected.

(c) According to the sixth axiom it suffices for us to prove that given an endomorphism

$$u : X_0 \to X_0$$

where $X_0 \in \mathcal{C}$ is connected, the image endomorphism under the fibre functor

$$F(u) : F(X_0) \to F(X_0)$$

is an isomorphism. But $F(X_0)$ is a finite set, so to prove that $F(u)$ is bijective it suffices to show that it is surjective. The third axiom gives us a splitting of u

$$\begin{tikzcd} X_0 \arrow[r, two heads, "u^{'}"] \arrow[rr, bend right, "u"'] & X_0^{'} \arrow[r, "u^{''}"] & X_0 = X_0^{'} \amalg X_0^{''} \end{tikzcd}$$

where $u^{'}$ is a strict epimorphism and $u^{''}$ is a monomorphism that induces an isomorphism onto the factor $X_0^{'}$. We assume that X_0 is connected so either we have $X_0^{'} = I_{\mathcal{C}}$ in which case $X_0 = I_{\mathcal{C}}$ and our claim is trivial, or $X_0 = X_0^{'}$ and $u^{''}$ is an isomorphism. Then u is an epimorphim and we are done.

□

Definition 4.14. From the above proposition 4.13, it follows that for if X_0 is a non-initial connected object in $\mathcal{C}$ and $\zeta_0 \in F(X_0)$ is any element, then the evaluation map

$$ev_{\zeta_0} : \mathrm{Aut}_{\mathcal{C}}(X_0) \hookrightarrow F(X_0)$$
$$\{u : X_0 \xrightarrow{\cong} X_0\} \mapsto F(u)(\zeta_0)$$

is injective. We call a connected $X_0 \in \mathcal{C}$ a *Galois object* if this evaluation map is bijective for all choices of $\zeta_0 \in F(X_0)$. This is is equivalent to the following:

(1) $\mathrm{Aut}_{\mathcal{C}}(X_0)$ acts transitively on $F(X_0)$,
(2) $\mathrm{Aut}_{\mathcal{C}}(X_0)$ acts simply transitively on $F(X_0)$,
(3) $|\mathrm{Aut}_{\mathcal{C}}(X_0)| = |F(X_0)|$,
(4) $X_0/\mathrm{Aut}_{\mathcal{C}}(X_0)$ is a final object in $\mathcal{C}$.

Remark 4.15. Notice that to show that an object X_0 is Galois, it is enough to show that the map

$$ev_{\zeta_0} : \mathrm{Aut}_{\mathcal{C}}(X_0) \hookrightarrow F(X_0)$$

is bijective for *some choice of* $\zeta_0 \in X_0$.

This definition might seem restrictive, but we now show that there is an abundance of Galois objects in any Galois category.

Proposition 4.16. *Let $X \in \mathcal{C}$ be a connected object. Then there exists a Galois object $\hat{X} \in \mathcal{C}$ such that*

$$\hat{X} \geq X$$

and $\hat{X}$ is minimal (w.r.t. the domination ordering) among the Galois objects that dominate X.

Proof. By 2, there exists an object $(X_0, \zeta_0) \in \mathcal{C}^{pt}$ such that $X_0 \in \mathcal{C}$ is connected and the evaluation map

$$ev_{\zeta_0} : \mathrm{Hom}_{\mathcal{C}}(X_0, X) \xrightarrow{\cong} F(X)$$

is bijective. In particular $\mathrm{Hom}_{\mathcal{C}}(X_0, X)$ is a finite set and we can write

$$\mathrm{Hom}_{\mathcal{C}}(X_0, X) = \{u_1, \ldots u_n\}.$$

For any $1 \leq i \leq n$ we denote $\zeta_i := F(u_i)(\zeta_0)$ and the canonical projections by

$$pr_i : X^n := X \times \ldots \times X \to X$$

The universal property of the product then gives us the existence of a map

$$\pi := (u_1, \ldots, u_n) : X_0 \to X^n$$

such that $pr_i \circ \pi = u_i$ for all i.

As before, the third axiom tells us that the map π can be decomposed

$$\begin{tikzcd} X_0 \arrow[r, two heads, "\pi'"] \arrow[rr, bend right, "\pi"'] & \hat{X} \arrow[r, "\pi''"] & X^n = \hat{X} \amalg \hat{X}' \end{tikzcd}$$

where π' is a strict epimorphism and π'' is a monomorphism that induces an isomorphism onto the factor $\hat{X}$.

Let us first check that this $\hat{X}$ that was uniquely defined by the decomposition of π, is a Galois object. It is the target of a strict epimorphism whose source is a connected object, thus 3b tells us that it is connected. Define

$$\hat{\zeta}_0 := F(\pi')(\zeta_0) = (\zeta_1, \ldots, \zeta_n) \in F(\hat{X})$$

We want to show that the evaluation map

$$ev_{\hat{\zeta}_0} : \mathrm{Aut}_{\mathcal{C}}(\hat{X}) \hookrightarrow F(\hat{X})$$

is surjective (and hence bijective). By 2 there exists some $(\tilde{X}_0, \tilde{\zeta}_0) \in \mathcal{C}^{pt}$ such that $\tilde{X}_0 \in \mathcal{C}$ is connected and such that for any $\zeta \in F(\hat{X})$ we have

$$(\tilde{X}_0, \tilde{\zeta}_0) \geq (X_0, \zeta_0)$$
$$(\tilde{X}_0, \tilde{\zeta}_0) \geq (\hat{X}, \zeta)$$

Therefore we may assume (up to replacing (X_0, ζ_0) by $(\tilde{X}_0, \tilde{\zeta}_0)$) that there exist morphisms

$$\rho_\zeta : (X_0, \zeta_0) \to (\hat{X}, \zeta)$$

in $\mathcal{C}^{pt}$. To show the surjectivity we need to show that for any $\zeta \in F(\hat{X})$ there exists an $\omega \in \mathrm{Aut}_{\mathcal{C}}(\hat{X})$ such that $F(\omega)(\hat{\zeta}_0) = \zeta$.

On the one hand, if we are given an $\omega \in \mathrm{Aut}_{\mathcal{C}}(\hat{X})$ we can write $F(\omega)(\hat{\zeta}_0) = F(\omega \circ \pi^{'})(\zeta_0)$ and on the other, if we are given some $\zeta \in F(\hat{X})$ we can write $\zeta = F(\rho_\zeta)(\zeta_0)$. We then see that given some $\zeta \in F(\hat{X})$ there exists an $\omega \in \mathrm{Aut}_{\mathcal{C}}(\hat{X})$ such that $F(\omega)(\hat{\zeta}_0) = \zeta$ if and only if

$$\omega \circ \pi^{'} = \rho_\zeta.$$

In order to prove that such an ω exists, we first notice that for each $1 \leq i \leq n$ the composition

$$pr_i \circ \pi^{''} \circ \rho_\zeta : X_0 \to X$$

is in $\mathrm{Hom}_{\mathcal{C}}(X_0, X)$ and so

$$\{pr_1 \circ \pi^{''} \circ \rho_\zeta, \ldots, pr_n \circ \pi^{''} \circ \rho_\zeta\} \subseteq \{u_1, \ldots, u_n\}.$$

If we prove that the maps $pr_i \circ \pi^{''} \circ \rho_\zeta$ are distinct, we have shown that

$$\{pr_1 \circ \pi^{''} \circ \rho_\zeta, \ldots, pr_n \circ \pi^{''} \circ \rho_\zeta\} = \{u_1, \ldots, u_n\}.$$

The u_i's are distinct so we notice that

$$pr_i \circ \pi^{''} \circ \pi^{'} = u_i \neq u_j = pr_j \circ \pi^{''} \circ \pi^{'}$$

when $i \neq j$ and the map $\pi^{'}$ is a strict epimorphism which implies that $pr_i \circ \pi^{''} \neq pr_j \circ \pi^{''}$. But $\rho_\zeta : X_0 \to \hat{X}$ is automatically a strict epimorphism by 3a since $\hat{X}$ is connected. Therefore

$$pr_i \circ \pi^{''} \circ \rho_\zeta \neq pr_j \circ \pi^{''} \circ \rho_\zeta$$

But since these $pr_i \circ \pi'' \circ \rho_\zeta$'s exhaust the set $\mathrm{Hom}_{\mathcal{C}}(X_0, X)$ and the map $pr_i \circ \pi'' \circ \pi'$ is in $\mathrm{Hom}_{\mathcal{C}}(X_0, X)$, there exists some $1 \leq \sigma(i) \leq n$ such that

$$pr_i \circ \pi'' \circ \rho_\zeta = pr_{\sigma(i)} \circ \pi'' \circ \pi'.$$

This defines a permutation $\sigma \in \mathcal{S}_n$ and the universal property of products tells us that it extends to a unique isomorphism $\Sigma : X^n \to X^n$ such that $pr_i \circ \Sigma = pr_{\sigma(i)}$. Thus we have

$$pr_i \circ \pi'' \circ \pi' = pr_i \circ \Sigma \circ \pi'' \circ \rho_\zeta$$

which forces the equality

$$\pi'' \circ \pi' = \Sigma \circ \pi'' \circ \rho_\zeta$$

We know that Σ is an isomorphism and thus $\Sigma \circ \pi''$ is a strict epimorphism. We also know that ρ_ζ is monic, so the uniqueness of the decomposition in the third axiom tells us that there exists an automorphism $\omega : \hat{X} \to \hat{X}$ such that

$$\Sigma \circ \pi'' = \pi'' \circ \omega$$
$$\omega \circ \pi' = \rho_\zeta$$

and the latter equality is precisely what we wanted to show.

What remains is to show that this $\hat{X}$ is minimal among the Galois objects in $\mathcal{C}$ that dominate X. Assume $Y \in \mathcal{C}$ is Galois and that we have a morphism $\phi : Y \to X$. We shall show that there exists a morphism $Y \to \hat{X}$ in $\mathcal{C}$. Now we fix for each $1 \leq i \leq n$ some $\eta_i \in F(Y)$ such that $F(\phi)(\eta_i) = \zeta_i$. The assumption that Y is Galois allows us to assume the existence for each $1 \leq i \leq n$ of an automorphism $\omega_i \in \mathrm{Aut}_{\mathcal{C}}(Y)$ such that $F(\omega_i)(\eta_1) = \eta_i$. This defines a unique morphism

$$\kappa = (\phi \circ \omega_1, \ldots, \phi \circ \omega_n) : Y \to X^n$$

such that $pr_i \circ \kappa = \phi \circ \omega_i$ for all i. As before we use the third axiom to give us a decomposition

$$\begin{tikzcd} Y \arrow[r, two heads, "\kappa'"] \arrow[rr, bend right, "\kappa"'] & Z' \arrow[r, "\kappa''"] & X^n = Z' \amalg Z'' \end{tikzcd}$$

where κ' is a strict epimorphism and κ'' is a monomorphism that induces an isomorphism onto the factor Z'. Then 3b assures us that since Z' is the target of a strict epimorphism from a connected object then Z' itself is a connected object. But

$$F(\kappa)(\eta_1) = (\zeta_1, \ldots, \zeta_n) = \hat{\zeta}_0$$

and hence Z' is the connected component of $\hat{\zeta}_0$ in X^n, i.e.

$$Z' = \hat{X}$$

and we have found our desired morphism from Y to $\hat{X}$ $\square$

We call this uniquely defined minimal Galois object $\hat{X}$ dominating X the *Galois closure of* X *in* $\mathcal{C}$.

We have now seen that for any object X in a Galois category $\mathcal{C}$ we can find a Galois object in $\mathcal{C}$ that dominates X. It is therefore natural to think that many properties of $\mathcal{C}$ (in particular some behaviour in the limits) can be investigated by studying these Galois objects.

We introduce some notation. Let $\mathcal{G}$ be a system of representatives of the isomorphism classes of Galois objects in $\mathcal{C}$ and fix

$$\underline{\zeta} = (\zeta_X)_{X \in \mathcal{G}} \in \prod_{X \in \mathcal{G}} F(X)$$

We now consider the subset of the pointed category $\mathcal{C}^{pt}$ consisting of elements (X, ζ_X) where $X \in \mathcal{G}$. An element (X, ζ_X) therein is said to dominate another element (Y, ζ_Y) if there exists a morphism $u_{X,Y} : X \to Y$ such that $F(u_{X,Y})(\zeta_X) = \zeta_Y$. Such a morphism must be unique and we can thus define a directed set in $\mathcal{C}^{pt}$ as

$$\mathcal{G}^{\underline{\zeta}} = \{(X, \zeta_X) \in \mathcal{C}^{pt} | X \in \mathcal{G}\}$$

For any such pair $(X, \zeta_X) \in \mathcal{G}^{\underline{\zeta}}$, we can look at the evaluation map

$$\begin{aligned} ev_{X,\zeta_X} : \mathrm{Hom}_{\mathcal{C}}(X, -) &\to F \\ \{u : X \to Y\} &\mapsto F(u)(\zeta_X) \end{aligned}$$

For any $X_0 \in \mathcal{G}$ we let $\mathcal{C}^{X_0} \subset \mathcal{C}$ denote the full subcategory whose objects are the $X \in \mathcal{C}$ whose connected components are all dominated by X_0.

Next we consider the Galois correspondence for Galois categories.

Proposition 4.17. *Given a Galois category $\mathcal{C}$ with a fibre functor F, and given some object $X_0 \in \mathcal{G}$ we have the following.*

(1) Restricting the evaluation map to $\mathcal{C}^{X_0}$ gives us an isomorphism of functors

$$ev_{X_0,\zeta_{X_0}} : \mathrm{Hom}_{\mathcal{C}}(X_0, -)|_{\mathcal{C}^{X_0}} \xrightarrow{\cong} F|_{\mathcal{C}^{X_0}}$$

and in particular it induces an isomorphism of groups

$$u^{\zeta_0} : \mathrm{Aut}(F|_{\mathcal{C}^{X_0}}) \xrightarrow{\cong} \mathrm{Aut}(\mathrm{Hom}_{\mathcal{C}}(X_0, -)|_{\mathcal{C}^{X_0}}) = \mathrm{Aut}_{\mathcal{C}}(X_0)^{op}.$$

(2) The functor

$$F|_{\mathcal{C}^{X_0}} : \mathcal{C}^{X_0} \to \boldsymbol{FinSet}$$

factors through $\mathcal{C}(\mathrm{Aut}_{\mathcal{C}}(X_0)^{op})$ *giving the commutative diagram*

$$\begin{tikzcd}
\mathcal{C}^{X_0} \arrow[r, "F|_{\mathcal{C}^{X_0}}"] \arrow[d, "\cong"', "F|_{\mathcal{C}^{X_0}}"] & \boldsymbol{FinSet} \\
\mathcal{C}(\mathrm{Aut}_{\mathcal{C}}(X_0)^{op}) \arrow[ur, "For"']
\end{tikzcd}$$

Proof. (1) Any morphism $u : Y \to X$ in $\mathcal{C}^{X_0}$ gives rise to a commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{C}}(X_0, Y) \arrow[r, "u\circ"] \arrow[d, "ev_{\zeta_0}(Y)"'] & \mathrm{Hom}_{\mathcal{C}}(X_0, X) \arrow[d, "ev_{\zeta_0}(X)"] \\
F(Y) \arrow[r, "F(u)"'] & F(X)
\end{tikzcd}$$

Therefore $ev_{X_0,\zeta_{X_0}}$ is a morphism of functors. Since X_0 is connected

$$ev_{\zeta_0}(X) : \mathrm{Hom}_{\mathcal{C}}(X_0, X) \hookrightarrow F(X)$$

is injectie for any $X \in \mathcal{C}^{X_0}$.

If we can show the case where X is connected, then the nonconnected case follows from the decomposition theorem and the fact that the fibre functor commutes with finite coproducts.

So let's assume X is connected. Then any morphism $u : X_0 \to X$ in $\mathcal{C}$ is automatically a strict epimorphism according to 3a. Write

$$F(X) = \{\zeta_1, \dots, \zeta_n\}$$

and let $\zeta_{0_i} \in F(X_0)$ be such that

$$F(u)(\zeta_{0_i}) = \zeta_i$$

for all $1 \le i \le n$. Since X_0 is a Galois object in $\mathcal{C}$ there exist $\omega_i \in \mathrm{Aut}_{\mathcal{C}}(X_0)$ such that $F(\omega_i)(\zeta_0) = \zeta_{0_i}$, from which we conclude that $ev_{X_0,\zeta_{X_0}}$ is surjective and thus a bijection.

(2) We denote

$$G := \mathrm{Aut}_{\mathcal{C}}(X_0).$$

We now identify the restricted fibre functor $F|_{\mathcal{C}^{X_0}}$ with the functor $\mathrm{Hom}_{\mathcal{C}}(X_0, -)|_{\mathcal{C}^{X_0}}$. We have a natural action of G^{op} on it by composing from the right, and by the definition of the equivalence of functors we see that it is G^{op}-equivariant. Therefore $F|_{\mathcal{C}^{X_0}}$ factors through $\mathcal{C}(G^{op})$ i.e.

$$\begin{tikzcd} \mathcal{C}^{X_0} \arrow[r, "F|_{\mathcal{C}^{X_0}}"] \arrow[d, "F|_{\mathcal{C}^{X_0}}"'] & \mathbf{FinSet} \\ \mathcal{C}(G^{op}) \arrow[ur, "For"'] & \end{tikzcd}$$

What remains is to show that

$$F|_{\mathcal{C}^{X_0}} : \mathcal{C}^{X_0} \to \mathcal{C}(G^{op})$$

is an equivalence of categories. We do that by showing that it is essentially surjective and fully faithful.

a) **Essential surjectivity:** Let S be a finite set with a G^{op} action, i.e. $S \in \mathcal{C}(G^{op})$ (we have a finite discrete group acting on a finite set so continuity is not an issue). We want to find an object X in $\mathcal{C}^{X_0}$ such that S is isomomorphic to $F(X)$. By the same argument as before we may assume that S is a connected object in $\mathcal{C}(G^{op})$. Connected objects in $\mathcal{C}(G^{op})$ are finite sets with a transitive G^{op}-action. Fix some element $s \in S$ and transitivity then gives us an epimorphism in $\mathcal{C}(G^{op})$

$$\begin{aligned} p_s^0 : G^{op} &\twoheadrightarrow S \\ g &\mapsto g \cdot s \end{aligned}$$

Now let

$$f_s := p_s^0 \circ ev_{\zeta_0}^{-1} : F(X_0) \twoheadrightarrow S.$$

Consider the stabilizer $S_s := \mathrm{Stab}_{G^{op}}(s)$ of s in G^{op} and any element $x \in S_s$ and any $\omega \in G$. Since we will be working with both G and G^{op} we shall use the symbol $\wedge$ for the composition law in G and $\vee$ for the composition law for G^{op}. Now

$$\begin{aligned} f_s \circ F(x)(ev_{\zeta_0}(\omega)) &= p_s^0 \circ ev_{\zeta_0^{-1}} \circ ev_{\zeta_0}(x \wedge \omega) \\ &= (x \wedge \omega) \cdot s \\ &= (\omega \vee x) \cdot s \\ &= \omega \cdot (x \cdot s) \\ &= \omega \cdot s \\ &= f_s(ev_{\zeta_0}(\omega)) \end{aligned}$$

so by the universal property of the quotient, $f_s : F(X_0) \twoheadrightarrow S$ factors as

$$\begin{tikzcd}
G^{op} \arrow[r, two heads, "ev_{\zeta_0}"] \arrow[dr, "p_s^0"'] & F(X_0) \arrow[r] \arrow[d, "f_s"] & F(X_0)/S_s \arrow[dl, "\exists! \overline{f_s}"] \\
& S &
\end{tikzcd}$$

By the second axiom the categorical quotient $p_s : X_0 \to X_0/S_s$ of X_0 by $S_s \subseteq G$ exists in $\mathcal{C}$, and by the fifth axiom the map $F(X_0) \twoheadrightarrow F(X_0)/S_s$ is

$$F(p_s) : F(X_0) \twoheadrightarrow F(X_0/S_s).$$

Now X_0 is by assumption connected so G^{op} acts transitively on $F(X_0)$ and therefore

$$|F(X_0)/S_s| = |F(X_0)|/|S_s| = [G : S_s] = |S|.$$

Then $\overline{f_s} : F(X_0)/S_s \twoheadrightarrow S$ is a surjective map of finite sets of equal cardinality and hence a bijection.

b) **Full faithfulness:** Given two objects X and Y in $\mathcal{C}^{X_0}$ we want to show that the functor $F|_{\mathcal{C}^{X_0}}$ induces a bijection

$$\begin{aligned}\mathrm{Hom}_{\mathcal{C}^{X_0}}(X, Y) =& \mathrm{Hom}_{\mathcal{C}}(X, Y)|_{\mathcal{C}^{X_0}} \to \\ & \mathrm{Hom}_{\mathcal{C}(G^{op})}(F|_{\mathcal{C}^{X_0}}(X), F|_{\mathcal{C}^{X_0}}(Y)).\end{aligned}$$

As before we may assume X and Y are connected objects. Then 1 immediately gives us the faithfulness, i.e. the injectivity of the map. Now let a morphism

$$u : F|_{\mathcal{C}^{X_0}}(X) \to F|_{\mathcal{C}^{X_0}}(Y)$$

be given in $\mathrm{Hom}_{\mathcal{C}(G^{op})}(F|_{\mathcal{C}^{X_0}}(X), F|_{\mathcal{C}^{X_0}}(Y))$ and fix an element $x \in F|_{\mathcal{C}^{X_0}}(X)$. Since u is a map in $\mathcal{C}(G^{op})$, i.e. it commutes with the G^{op} action, we have $S_x \subseteq S_{u(x)}$ and therefore $p_{u(x)} : X_0 \to X_0/S_{u(x)}$ factors through X_0/S_x:

$$\begin{tikzcd}
X_0 \arrow[r, "p_x"] \arrow[d, "p_{u(x)}"'] & X_0/S_x \arrow[dl, "p_{x,u(x)}"] \\
X_0/S_{u(x)} &
\end{tikzcd}$$

So the same argument as we used to prove essential surjectivity gives us the following commutative diagram

$$\begin{tikzcd}
 & F(X_0) \arrow[dl, "F(p_x)"'] \arrow[dr, "F(p_{u(x)})"] & \\
F(X_0/S_x) \arrow[rr, "F(p_{x,u(x)})"] \arrow[d, "\bar{f}_x"', "\cong"] & & F(X_0/S_{u(x)}) \arrow[d, "\cong"', "\bar{f}_{u(x)}"] \\
F(X) \arrow[rr, "u"'] & & F(Y)
\end{tikzcd}$$

from which we see that there exists a morphism $X \to Y$ that $F|_{\mathcal{C}^{X_0}}$ maps to u, proving the the surjectivity of the induced map i.e. the fullness of the functor $F|_{\mathcal{C}^{X_0}}$.

□

As we noted before $\mathcal{G}^{\underline{\zeta}}$ is a directed set, and if $u_{X,Y} : X \to Y$ is the unique morphisms such that $F(u_{X,Y})(\zeta_X) = \zeta_Y$ then the proposition we just proved gives us a commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{C}}(X,-) \arrow[r, "ev_{X,\zeta_X}"] \arrow[d, "u^*_{X,Y}"] & F(-) \\
\mathrm{Hom}_{\mathcal{C}}(Y,-) \arrow[ur, "ev_{Y,\zeta_Y}"'] &
\end{tikzcd}$$

which allows us to pass to the limit and obtain a morphism of functors

$$ev_{\underline{\zeta}} : \varinjlim_{\mathcal{G}^{\underline{\zeta}}} \mathrm{Hom}_{\mathcal{C}}(X,-) \to F(-)$$

We have the following proposition

Proposition 4.18. *The morphism $ev_{\underline{\zeta}}$ is an isomorphism.*

Proof. The morphism $u_{X,Y}$ is a strict epimorphism and so the result follows from the isomorphism

$$ev_{X,\zeta_X} : \mathrm{Hom}_{\mathcal{C}}(X,-)|_{\mathcal{C}^X} \to F|_{\mathcal{C}^X}$$

obtained in the previous proposition. □

We now define pro-objects and pro-representable functors.

Definition 4.19. With any category $\mathcal{D}$ we can associate another category $Pro(\mathcal{D})$. The objects of $Pro(\mathcal{D})$ are projective systems

$$\underline{X} := (\phi_{i,j} : X_i \to X_j)_{i,j\in I\, i\geq j}$$

indexed by directed posets $(I, \leq)$. Morphisms in $Pro(\mathcal{D})$ from $\underline{X} = (\phi_{i,j} : X_i \to X_j)$ to $\underline{X}' = (\phi'_{i',j'} : X'_{i'} \to X'_{j'})$ are given by

$$\mathrm{Hom}_{Pro(\mathcal{D})}(\underline{X}, \underline{X}') = \varprojlim_{i' \in I'} \varinjlim_{i \in I} \mathrm{Hom}_{\mathcal{C}}(X_i, X'_{i'}).$$

By looking at one point index sets we can canonically identify $\mathcal{D}$ with a full subcategory of $Pro(\mathcal{D})$ and any functor $G : \mathcal{D} \to \mathbf{FinSet}$ extends canonically to a functor

$$Pro(G) : Pro(\mathcal{D}) \to Pro(\mathbf{FinSet}).$$

Definition 4.20. Let $\mathcal{D}$ be a category and $G : \mathcal{D} \to \mathbf{FinSet}$ be a functor. We say that G is *pro-representable in* $\mathcal{D}$ if there exists some pro-object $\underline{X} = (\phi_{i,j} : X_i \to X_j)$ such that we have an isomorphism of functors

$$\mathrm{Hom}_{Pro(\mathcal{D})}(\underline{X}, -)|_{\mathcal{D}} \xrightarrow{\cong} G$$

We say that G is *strictly pro-representable in* $\mathcal{D}$ if it is pro-representable by a pro-object $\underline{X} = (\phi_{i,j} : X_i \to X_j)$ all of whose transition morphisms $\phi_{i,j}$ are epimorphisms.

The previous proposition 4.18, along with the fact that morphisms here are strict epimorphisms because we are dealing with Galois (and hence connected) objects can then be restated as:

*If $\mathcal{C}$ is a Galois category and $F : \mathcal{C} \to$ **FinSet** is a fibre functor, then F is strictly pro-representable in $\mathcal{C}$ by the object $\mathcal{G}^{\leq} \in Pro(\mathcal{C})$.*

Now we are edging closer to the main theorems of this chapter but first we look at a lemma and a proposition.

Lemma 4.21. *For any two Galois objects $X, Y \in \mathcal{G}$ with $Y \geq X$, any morphisms $\phi, \psi : Y \to X$ in $\mathcal{C}$ and for any automorphism $\omega_Y \in \mathrm{Aut}_{\mathcal{C}}(Y)$ there exists a unique automorphism $r_{\phi,\psi} \in \mathrm{Aut}_{\mathcal{C}}(X)$ making the diagram*

$$\begin{tikzcd} Y \arrow[r, "\omega_Y"] \arrow[d, "\psi"'] & Y \arrow[d, "\phi"] \\ X \arrow[r, "r_{\phi,\psi}"] & X \end{tikzcd}$$

commute.

Proof. X is by assumption Galois and therefore connected, so $\psi : Y \to X$ is a strict epimorphism. By the definition of strict epimorphisms, composition by ψ gives an injective map

$$\circ\psi : \mathrm{Aut}_{\mathcal{C}}(X) \hookrightarrow \mathrm{Hom}_{\mathcal{C}}(Y, X).$$

Again, since X is Galois we have

$$|\mathrm{Hom}_{\mathcal{C}}(Y,X)| = |\mathrm{Aut}_{\mathcal{C}}(X)| = |F(X)|.$$

Therefore the map $\circ\psi : \mathrm{Aut}_{\mathcal{C}}(X) \to \mathrm{Hom}_{\mathcal{C}}(Y,X)$ is bijective and in particular there is a unique $r_{\phi,\psi} \in \mathrm{Aut}_{\mathcal{C}}(X)$ making the above diagram commutative. □

Proposition 4.22. *We have an isomorphism of profinite groups*

$$\pi_1(\mathcal{C},F) \cong \varprojlim_{\mathcal{G}^{\underline{\zeta}}} \mathrm{Aut}_{\mathcal{C}}(X)^{op}$$

Proof. By 4.21 we get a well defined surjective map

$$r_{\phi,\psi} : \mathrm{Aut}_{\mathcal{C}}(Y) \twoheadrightarrow \mathrm{Aut}_{\mathcal{C}}(X)$$

which is a group epimorphism when $\phi = \psi$. Therefore we get an inverse system of finite groups. Denote by Π the inverse limit of the system and then we have an action of Π^{op} on $\mathrm{Hom}_{\mathcal{C}}(X,-)$ by composition on the right. This induces a group monomorphism

$$\Pi^{op} \hookrightarrow \mathrm{Aut}\left(\varinjlim_{\mathcal{G}^{\underline{\zeta}}} \mathrm{Hom}_{\mathcal{C}}(X,-)\right)$$

Recall that we have an isomorphism of functors

$$ev_{\underline{\zeta}} : \varinjlim \mathrm{Hom}_{\mathcal{C}}(X,-)|_{\mathcal{C}} \xrightarrow{\cong} F$$

and combining this with the group monomorphism we obtain a group monomorphism

$$\begin{aligned} u^{\underline{\zeta}} : \pi_1(\mathcal{C},F) &\hookrightarrow \Pi^{op} \\ \theta &\mapsto (ev_{\zeta_X}^{-1}(\theta(X)(\zeta_X)))_{X\in\mathcal{G}} \end{aligned}$$

Now we first prove that this is an isomorphism of groups and then that it is a continuous isomorphism with a continuous inverse.

Let us construct an inverse. Take any

$$\underline{\omega} := (\omega_X)_{X\in\mathcal{G}} \in \Pi.$$

If $Z \in \mathcal{C}$ is connected, let us denote by $\hat{Z}$ the Galois closure of Z in $\mathcal{C}$. We have a bijective map

$$\begin{aligned} \theta_{\underline{\omega}}(Z) : F(Z) &\xrightarrow{ev_{\zeta_{\hat{Z}}}^{-1}} \mathrm{Hom}_{\mathcal{C}}(\hat{Z},Z) \\ &\xrightarrow{\circ\omega_{\hat{Z}}} \mathrm{Hom}_{\mathcal{C}}(\hat{Z},Z) \\ &\xrightarrow{ev_{\zeta_{\hat{Z}}}} F(Z) \end{aligned}$$

This clearly gives us an automorphism of the functor F and

$$u^{\underline{\zeta}}(\theta_{\underline{\omega}}) = \underline{\omega}$$

To show that $u^{\underline{\zeta}}$ is continuous it is enough to show that for any $X \in \mathcal{G}$ the composition with the canonical map

$$\pi_1(\mathcal{C}, F) \to \Pi^{op} \to \mathrm{Aut}_{\mathcal{C}}(X)^{op}$$

is continuous. This is clear from the definition of the topology on $\pi_1(\mathcal{C}, F)$. Finally we notice that $\pi_1(\mathcal{C}, F)$ is compact and Π^{op} is Hausdorff, so this continous group isomorphism is automatically an isomorphism of topological groups. □

Finally we come to the main theorems.

Theorem 4.23. *Let $\mathcal{C}$ be a Galois category with a fiber functor F. Then the fibre functor induces an equivalence of categories*

$$\mathcal{C} \xrightarrow{\cong} \pi_1(\mathcal{C}, F) - \mathrm{Rep_c}$$

Proof. The fibre functor F is pro-representable by $\mathcal{G}^{\underline{\zeta}}$, and $\pi_1(\mathcal{C}, F)$ is isomorphic to Π^{op} as profinite groups, so this amounts to showing that

$$\mathrm{Hom}_{Pro(\mathcal{C})}(\mathcal{G}^{\underline{\zeta}}, -)|_{\mathcal{C}} : \mathcal{C} \to \mathbf{FinSet}$$

factors through an equivalence

$$F^{\underline{\zeta}} : \mathcal{C} \to \Pi^{op} - \mathrm{Rep_c}.$$

We check that $F^{\underline{\zeta}}$ is essentially surjective and fully faithful:

- Take any $E \in \Pi^{op} - \mathrm{Rep_c}$. Since E has the discrete topology there exists some Galois object $X \in \mathcal{G}$ such that the action of Π^{op} factors through the finite quotient $\mathrm{Aut}_{\mathcal{C}}(X)$ and by the Galois correspondence in $\mathcal{C}^X$ this shows that $F^{\underline{\zeta}}$ is essentially surjective.
- Let any $Y, Y^{'} \in \mathcal{C}$ be given. Then there exists a Galois object X that dominates them both,

  $$X \geq Z,\ X \geq Z^{'}$$

  This allows us to use the Galois correspondence in $\mathcal{C}^X$ to see that $F^{\underline{\zeta}}$ is fully faithful.

□

Definition 4.24. Let $\mathcal{C}$ be a Galois category and let F_1 and F_2 be two fibre functors for $\mathcal{C}$. The set of isomorphisms of functors $F_1 \to F_2$ is called the set of paths from F_1 to F_2 and is denoted by $\pi_1(\mathcal{C}; F_1, F_2)$.

Theorem 4.25. *Let $\mathcal{C}$ be a Galois category and let F_1 and F_2 be fibre functors for $\mathcal{C}$. Then $\pi_1(\mathcal{C}; F_1, F_2)$ is non-empty and there is a non-canonical isomorphism*

$$\pi_1(\mathcal{C}, F_1) \cong \pi_1(\mathcal{C}, F_2)$$

Proof. By 4.18 the fibre functors are strictly pro-representable (see discussion after 4.20) so for any $\underline{\zeta}^i \in \prod_{X \in \mathcal{G}} F_i(X)$, $i = 1, 2$ we have an isomorphism of functors

$$ev_{\underline{\zeta}^i} : \mathrm{Hom}_{Pro(\mathcal{C})}(\mathcal{G}^{\underline{\zeta}^i}, -)$$

It is therefore enough to prove that

$$\mathcal{G}^{\underline{\zeta}^1} \cong \mathcal{G}^{\underline{\zeta}^2}$$

in $Pro(\mathcal{C})$.

Now by the Galois correspondence 4.17 we have

$$\varprojlim_X \varinjlim_Y \mathrm{Hom}_{\mathcal{C}}(Y, X) = \varprojlim_X \varinjlim_Y \mathrm{Aut}_{\mathcal{C}}(X)$$

where $X \in \mathcal{G}^{\underline{\zeta}^1}$ and $Y \in \mathcal{G}^{\underline{\zeta}^2}$. But we furthermore notice that the right hand side of the equation does not depend on Y so we have

$$\varprojlim_X \varinjlim_Y \mathrm{Aut}_{\mathcal{C}}(X) = \varprojlim_X \mathrm{Aut}_{\mathcal{C}}(X)$$

Now the projective limit of a system of finite non-empty sets is non-empty, so we see that there exists a morphism of pro-objects

$$f : \mathcal{G}^{\underline{\zeta}^1} \to \mathcal{G}^{\underline{\zeta}^2}$$

symmetrically we get the existence of a morphism of pro-objects

$$g : \mathcal{G}^{\underline{\zeta}^2} \to \mathcal{G}^{\underline{\zeta}^1}$$

and thus an endomorphism

$$g \circ f : \mathcal{G}^{\underline{\zeta}^1} \to \mathcal{G}^{\underline{\zeta}^1}$$

From 1 we see that the only endomorphism of $\mathcal{G}^{\underline{\zeta}^1}$ is the identity, proving that f and g are inverse to each other. $\square$

4.2. **Topological fundamental groups and infinite Galois theory in light of Galois categories.** We first reframe the infinite Galois theory we looked at in 2 in terms of Galois categories.

Example 4.26. We saw in 2.18 that if k is a field with a fixed choice of an algebraic closure $\bar{k}$ and $\mathbf{F\acute{E}t}_k$ denotes the category of finite dimensional étale algebras over k then the functor

$$\mathrm{Hom}_k(-, k_s) : (\mathbf{F\acute{E}t}_k)^{op} \to \mathrm{Gal}(k) - \mathrm{Rep_c}$$

is an equivalence of categories, where k_s denotes the seperable closure of k in $\bar{k}$. Thus we know that since $\mathrm{Gal}(k)$ is profinite (2.8) that $(\mathbf{F\acute{E}t}_k)^{op}$ is a Galois category with fiber functor $\mathrm{Hom}_k(-, k_s)$

Now we can reframe the theory of covering spaces and fundamental groups from 3 in terms of Galois categories.

Example 4.27. In 3.19 we saw that if X is a connected, locally path-connected and semilocally connected topological space and x is some point in X, then the functor

$$\mathrm{Fib}_x : f\mathrm{Cov}(X) \to \widehat{\pi_1(X, x)} - \mathrm{Rep_c}$$

is an equivalence of categories. Therefore $f\mathrm{Cov}(X)$ is a Galois category with fibre functor Fib_x and

$$\pi_1(f\mathrm{Cov}(X), \mathrm{Fib}_x) = \widehat{\pi_1(X, x)}$$

Furthermore we reframed this in terms of locally constant sheaves and saw in 3.23 that the functor

$$\begin{aligned} \mathrm{Sta}_x : f\mathrm{Loc}(X) &\to \widehat{\pi_1(X, x)} - \mathrm{Rep_c} \\ \mathcal{L} &\mapsto \mathcal{L}_x \end{aligned}$$

is an equivalence, and hence that $f\mathrm{Loc}(X)$ is a Galois category with fibre functor Sta_x and

$$\pi_1(f\mathrm{Cov}(X), \mathrm{Sta}_x) = \widehat{\pi_1(X, x)}$$

5. Étale covers and the étale fundamental group

Throughout this chapter we assume that we have a base scheme X that is connected and locally Noetherian. Many of the results presented here hold for schemes that are not locally Noetherian but others do not. The results that hold in general are usually more easy to prove in the case where we assume our schemes are locally notherian.

5.1. **Étale morphisms of schemes.**

We start by looking at some finiteness properties of morphisms of schemes.

Definition 5.1. A morphism of schemes $f : Y \to X$ is called *finite* if there exists a covering of X by open affine subschemes $U_i = \operatorname{Spec} A_i$, such that $f^{-1}(U_i)$ is affine for all i, say $f^{-1}(U_i) = \operatorname{Spec} B_i$ and the induced $A_i - Mod$ structure on B_i makes it a finitely generated A_i–module.

Definition 5.2. A morphism $f : Y \to X$ of schemes is called *locally of finite type* if there exist coverings $\{V_i\}$ of X and

$$f^{-1}(V_i) = \cup_{j \in I_j} U_j$$

of affine open subschemes such that for all i, j the induced ring map

$$\mathcal{O}_X(V_i) \to \mathcal{O}_Y(U_j)$$

endows $\mathcal{O}_Y(U_j)$ with the structure of a finitely generated $\mathcal{O}_X(V_i)$ algebra.

We say that $f : Y \to X$ is of finite type if it is locally of finite type and quasi-compact.

Remark 5.3. Notice that the property of being a finite morphism is much stronger then being locally of finite type. For example $\mathbb{C}[x]$ is a finitely generated $\mathbb{C}$-algebra but an infinite dimensional $\mathbb{C}$-module.

There is another finiteness property for morphism that is stronger then being locally of finite type.

Definition 5.4. A morphism of schemes $f : Y \to X$ is said to be locally of finite presentation if for any $y \in Y$ there exists an open affine $U \ni f(y)$ and an open affine $V \ni y$ such that $f(V) \subseteq U$ and $\mathcal{O}_Y(U)$ is a finitely presented $\mathcal{O}_X(V)$-algebra. Recall that if $A \to B$ is an A-algebra, then B is said to be finitely presented if it is isomorphic to an A-algebra of the form

$$B \cong A[T_1, \ldots, T_n]/K$$

where K is a finitely generated A-algebra.

We made the assumption at the beginning of this chapter that we were working with locally Noetherian schemes. This allows us to simplify a bit.

Proposition 5.5. *Let $f : Y \to X$ be a morphism locally of finite type such that Y is locally Noetherian. Then f is locally of finite presentation.*

Proof. See for example Stacks Project [2017, Tag 06G4]. □

The following lemma is crucial since we are going to define coverings as surjective finite étale morphisms and the fibre functors will be precisely the functor of taking fibres over a some (geometric point) and they have to be finite.

Lemma 5.6. *Let $f : Y \to X$ be a finite morphism of schemes. For each point $x \in X$ the fibre $f^{-1}(x)$ is a finite discrete subscheme of Y.*

Proof. Stacks Project [2017, Tag 02NU] tells us that finite morphisms are quasi-finite, and Stacks Project [2017, Tag 02NH] tells us that quasi-finite morphisms have finite discrete fibres. □

Remark 5.7. Herein lies the major difference between finite morphisms and morphisms of finite type, namely that finite morphisms have finite fibres and morphisms of finite type have *finite dimensional* fibres.

Lemma 5.8. *Finite morphisms are closed.*

Proof. See for example Görtz and Wedhorn [2010, Prop. 12.12] □

Lemma 5.9. *Any closed immersion is finite.*

Proof. See Stacks Project [2017, Tag 035C] □

In the following proposition we list some standard properties of finite morphisms, morphisms locally of finite type and morphisms locally of finite presentation.

Proposition 5.10. *We let $f : Y \to X$ be a morphism of schemes with the property $\mathbb{P}$ which is either, finite, or locally of finite type or locally of finite presentation. Then the following holds.*

(1) (Composition) *If $g : Z \to Y$ is another morphism with property $\mathbb{P}$, then the composition $f \circ g : Z \to X$ has property $\mathbb{P}$.*

(2) (Base change) *If $g : Z \to X$ is any morphism then the base change map*

$$p_2 : Y \times_X Z \to Z$$

has property $\mathbb{P}$.

Proof. See Stacks Project [2017, Tag 01T1, lemmas 28.14.3 and 28.14.4] for morphisms locally of finite type, Stacks Project [2017, Tag 01T0, lemmas 28.20.3 and 28.20.4] for morphisms locally of finite presentation and Stacks Project [2017, Tag 01WG, lemmas 28.42.5 and 28.42.6] for finite morphisms. □

Now we recall the definitions and some basic properties of Kähler differentials.

Definition 5.11. Let $A \to B$ be an A-algebra and M be a B-module. Then an A-derivation of B into M is a map

$$d : B \to M$$

such that

(1) d is additive
(2) $d(a) = 0$ for all $a \in A$
(3) $d(b_1 b_1) = b_1 d(b_2) + b_2 d(b_1)$

Definition 5.12. Let $A \to B$ be an A-algebra. The module of Kähler differentials (or relative differential forms) of B over A is a pair

$$(\Omega^1_{B/A}, d)$$

where $\Omega^1_{B/A}$ is a B-module and

$$d : B \to \Omega^1_{B/A}$$

is an A-derivation satisfying the follwing universal property:

For any B-module M and for any A-derivation $d^{'} : B \to M$ there exists a unique homomorphism of B-modules

$$\phi : \Omega^1_{B/A} \to M$$

such that $d^{'}$ factors through ϕ, i.e. such that the following diagram commutes

$$\begin{tikzcd} B \arrow[r, "d"] \arrow[dr, "d^{'}"'] & \Omega^1_{B/A} \arrow[d, "\phi"] \\ & M \end{tikzcd}$$

The basic question to ask now is whether this $\Omega^1_{B/A}$ exists (we shall often omit mentioning the derivation d and talk about "the module of Kähler differentials" simply as $\Omega^1_{B/A}$).

As often is the case, we prove the existence of such a universal object by constructing it and then proving that it satisfies the properties that we want.

Proposition 5.13. *For any ring A and any A-algebra $A \to B$ the module of Kähler differentials $(\Omega^1_{B/A}, d)$ exists and is unique up to unique isomorphism.*

Proof. If we assume existence then the proof of the uniquness is the standard one when dealing with universal objects and we shall not concern ourselves with it here.
As for the existence, we let F be the free B-module on all symbols db

where $b \in B$. Then consider the module

$$\Omega^1_{B/A} = F/\text{relations}$$

where the relations are

(1) $d(b_1 + b_2) - b_1 - b_2 = 0$
(2) $da = 0$
(3) $d(b_1 b_2) - b_1 db_2 - b_2 db_1 = 0$

for all $b_1, b_2 \in B$ and $a \in A$.

The relations chosen suggest the natural choice of a derivation

$$d : B \to \Omega^1_{B/A}$$
$$b \mapsto [db]$$

where $[db]$ is the class of $db\ inF$ in the quotient module (and will be denoted, by abuse of notation, by db). It is then routine to check that this pair $(\Omega^1_{B/A}, d)$ satisfies the universal properties of the module of Kähler differentials. □

Before we introduce étale algebras we look at some properties of Kähler differentials.

The first one we look at is how they behave under base change.

Proposition 5.14. *Let $A \to B$ and $A \to A'$ be an A-algebra and set*

$$B' := B \otimes_A A'$$

Then there exists a canonical isomorphism of B'-modules

$$\Omega^1_{B'/A'} \cong \Omega^1_{B/A} \otimes_B B'$$

Proof. See for example Liu [2006, Chapter 6, Prop. 1.8.(a)]. □

There is an obvious corollary to 5.14 that is useful later on and worth stating explicitly.

Corollary 5.15. *As in 5.14 we let $A \to B$ and $A \to A'$ be an A-algebra and set*

$$B' := B \otimes_A A'$$

Then if

$$\Omega^1_{B/A} = 0$$

then

$$\Omega^1_{B'/A'} = 0$$

Secondly we look at the two fundamental exact sequences concerning Kähler differentials.

Proposition 5.16 (First fundamental exact sequence)**.** *Let $\phi_B : A \to B$ and $\phi_C : A \to C$ be A-algebras. Let $\psi : B \to C$ be a homomorphism of A-algebras. Then we have an exact sequence*

$$\Omega^1_{B/A} \otimes_B C \xrightarrow{\alpha} \Omega^1_{C/A} \xrightarrow{\beta} \Omega^1_{C/B} \longrightarrow 0$$

where α and β are the maps defined as follows.

(1) We have a naturally defined bilinear map of C-modules

$$\gamma : \Omega^1_{B/A} \times C \to \Omega^1_{C/A}$$

given by

$$\gamma((d(b), c)) = cd(\rho(b))$$

This factors through the tensor product, and the resulting uniquely defined map from $\Omega^1_{B/A} \otimes_B C$ to $\Omega^1_{C/A}$ is the map

$$\alpha : \Omega^1_{B/A} \otimes_B C \to \Omega^1_{C/A}$$

Notice that α is explicitly given by

$$\alpha(d_{B/A}(b) \otimes c) = cd_{C/A}(\psi(b))$$

(2) If we let M be a C-module, then any B-derivation

$$\delta : C \to M$$

is by restriction of scalars an A-derivation. In particular, by choosing $M = \Omega^1_{C/B}$ and $\delta = d_B$ where $d_{C/B}$ is the universal B-derivation $C \to \Omega^1_{C/B}$ we see that the universal property for A-derivations gives us a unique map

$$\beta : \Omega^1_{C/A} \to \Omega^1_{C/B}$$

Explicitly β is simply given by

$$\beta(d_{C/A}(c)) = d_{C/B}(c)$$

Proof. See Matsumura [1980, Theorem 57]. $\square$

We have the following very useful corollary of the first exact sequence 5.16.

Corollary 5.17. *As before we let $A \to B$ be an A-algebra. Now let S be a multiplicative subset of B. Then*

$$S^{-1}\Omega^1_{B/A} \cong \Omega_{S^{-1}B/A}$$

Proposition 5.18 (Second fundamental exact sequence)**.** *Let $A \to B$ be an A-algebra, let $I \subset B$ be an ideal of B and let $C = B/I$. Then we have an exact sequence*

$$I/I^2 \xrightarrow{\delta} \Omega^1_{B/A} \otimes_B C \xrightarrow{\alpha} \Omega^1_{C/A} \longrightarrow 0$$

where the map δ is given by

$$\delta(\bar{b}) = db \otimes 1$$

for $b \in B$ (more precisely: for $b \in I \subset B$) and $\bar{b}$ the image of b in I/I^2. And α is the map we defined in the first exact ssequence 5.16

Proof. See Matsumura [1980, Theorem 58] □

Definition 5.19. We say that a homomorphism of rings $f : A \to B$ is *unramified* if B is finitely generated as an A-module and $\Omega^1_{B/A} = 0$

Example 5.20. The most basic examples of such unramified morphisms comes when $B = A/I$ for some (finitely generated) ideal $I \subset A$ or when $B = S^{-1}A$ where S is some multiplicatively closed subset of A.

The following proposition gives us an equivalent formulation for unramified morphisms.

Proposition 5.21. *Let $A \to B$ be a finitely presented A-algebra. Then the following are equivalent.*

(1) B is an unramified A-algebra.
(2) For any prime ideal $\mathfrak{q} \subset B$ we let $\mathfrak{p} := A \cap \mathfrak{q}$. Then the natural map

$$\kappa(\mathfrak{p}) : A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}} \to B_{\mathfrak{q}}/\mathfrak{q}B_{\mathfrak{q}}$$

is a seperable field extension.

Proof. Consider SGA1, Prop. 3.1 for the affine schemes $X = \mathrm{Spec}\,(B)$ and $Y = \mathrm{Spec}\,(A)$. □

Now we can define étale algebras.

Definition 5.22. An A-algebra $A \to B$ is said to be an *étale A-algebra* if it is flat and unramified

So an A-algebra $A \to B$ is étale if

(1) It is finitely presented.
(2) It is flat.
(3) $\Omega^1_{B/A} = 0$.

Remark 5.23. In 2.13 we had the definition of a finite étale k-algebra, as follows. A finite dimensional k-algebra A is finite étale if it is the direct product of finitely many seperable field extensions of k

$$A \cong \prod k_i$$

For a finite dimensional k-algebra A this is equivalent to our new definition of étale algebras. See Szamuely [2009, Prop. 5.1.31].

There is an equivalent formulation of étale algebras, known as the Jacobian criterion, that really underlines the analogy between étale morphisms in algebraic geometry and local isomorphisms in differential geometry.

Proposition 5.24. *Let $A \to B$ be an A-algebra. Then B is an étale A-algebra if and only if there exists a presentation*

$$B = A[x_1, \dots, x_n]/(f_1, \dots, f_m)$$

such that the Jacobian

$$\det\left(\frac{\partial f_i}{\partial x_j}\right)$$

is a unit in B.

Proof. See Milne [1980, Cor. 3.16]. □

The theory of modules of Kähler differentials has a sheaf-theoretic counterpart.

We have the following proposition-definition.

Proposition 5.25. *Let $f : Y \to X$ be a morphism of schemes. Then there exists a unique quasi-coherent sheaf $\Omega^1_{Y/X}$ on Y such that for any affine open subset V of X, any affine open subset U of $f^{-1}(V)$ and any $x \in U$ we have*

$$\Omega^1_{Y/X}|_U \cong (\Omega_{\mathcal{O}_Y(U)/\mathcal{O}_X(V)})^{\sim}$$
$$(\Omega^1_{Y/X})_x \cong \Omega_{\mathcal{O}_{Y,y}/\mathcal{O}_{X,f(y)}}$$

We call this quasi-coherent sheaf the sheaf of Kähler differentials (or relative differentials) of degree 1 of Y over X.

Proof. See Liu [2006, Cha. 6, Prop. 1.17]. □

Another equivalent way to define $\Omega^1_{Y/X}$ is given by the following proposition(see Hartshorne [1977, Cha. II.8, rem. 8.9.2] and Liu [2006, Cha. 6, rem. 1.18] for the equivalence).

We let $f : Y \to X$ be a morphism of schemes and we consider the diagonal morphism

$$\Delta : Y \to Y \times_X Y$$

associated with f. This gives us an isomorphism of Y to its image $\Delta(Y)$ in $Y \times_X Y$. This image is a locally closed subscheme in the fibre product, i.e. its a closed subscheme of an open subset $W \subseteq Y \times_X Y$.

Proposition 5.26. *If $\mathcal{I}$ is the sheaf of ideals of $\Delta(Y)$ in W. Then*

$$\Omega^1_{Y/X} = \Delta^*(\mathcal{I}/\mathcal{I}^2)$$

The three important propositions on modules of Kähler differentials, namely 5.14, 5.16 and 5.18, all have their sheaf theoretic counterparts. Their veracity follows directly from their algebraic versions. First of all we have nice behavior of the Kähler differentials under base-change.

Proposition 5.27. *Let $f : Y \to X$ be a morphism of schemes and let X' be an X-scheme. Define $Y' := Y \times_X X'$ and denote by $p : Y' \to Y$ the first projection. Then*

$$\Omega^1_{Y'/X'} \cong p^*\Omega^1_{Y/X}$$

We also have the *first fundamental exact sequence* for the sheaves of Kähler differentials.

Proposition 5.28. *Let $f : Y \to X$ and $X \to Z$ be morphisms of schemes. Then the following sequence of $\mathcal{O}_Y$-modules is exact.*

$$f^*\Omega^1_{X/Z} \to \Omega^1_{Y/Z} \to \Omega^1_{Y/X} \to 0$$

And similarly we have the *second fundamental exact sequence* for the sheaves of Kähler differentials.

Proposition 5.29. *Let $f : Y \to X$ be a morphism of schemes and let $Z \subseteq Y$ be a closed subscheme defined by the ideal sheaf $\mathcal{I}$. Then the following sequence is exact.*

$$\mathcal{I}/\mathcal{I}^2 \xrightarrow{\delta} \Omega^1_{Y/X} \otimes_{\mathcal{O}_Y} \mathcal{O}_Z \longrightarrow \Omega^1_{Z/X} \longrightarrow 0$$

Finally we look at how the restriction morphisms look for $\Omega^1_{Y/X}$.

Proposition 5.30. *Let $f : Y \to X$ be a morphism of schemes and let $U \subseteq Y$ be an open subset. Then*

$$\Omega^1_{Y/X}|_U \cong \Omega^1_{U/X}$$

Proof. This follows directly from the definiton 5.25. □

We can now define unramified morphisms of schemes.

Definition 5.31. A morphism of schemes $f : Y \to X$ is said to be unramified if it is locally of finite presentation and

$$\Omega^1_{Y/X} = 0$$

Remark 5.32. This is equivalent to saying that for any point $y \in Y$ there exists an affine open neighborhood $\operatorname{Spec} B = U \subset Y$ of y and an affine open neighborhood $\operatorname{Spec} A = V \subset X$ of $f(y)$ such that $f(U) \subset V$ and the induced map $A \to B$ is an unramified map of rings.

This notion of being unramified behaves well under composition and base-change.

Proposition 5.33. *We let $f : Y \to X$ be an unramified morphism of schemes.*

(1) (Composition) *If $g : Z \to Y$ is another unramified morphism, then the composition $f \circ g : Z \to X$ is unramified.*
(2) (Base change) *If $g : Z \to X$ is any morphism then the base change map*

$$p_2 : Y \times_X Z \to Z$$

is an unramified morphism.

Proof. This follows directly from 5.28 and 5.27. □

As schemes are Hausdorff only in the most trivial circumstances, it is not a useful notion in algebraic geometry. There is however an algebraic analogue of Hausdorffness, namely separatedness.

Definition 5.34. A morphism $f : Y \to X$ is said to be separated if the induced diagonal map

$$\Delta_{Y/X} : Y \to Y \times_X Y$$

is a closed immersion. We say that Y is separated over X.

One can easily see how this is related to the Hausdorff property by noticing that a topological space X is Hausdorff if and only if the diagonal is a closed subset of $X \times X$.

Definition 5.35. A morphism $f : Y \to X$ of schemes is said to be *flat at* $y \in Y$ if the induced homomorphism of local rings $f_x^\sharp : \mathcal{O}_{X,f(y)} \to \mathcal{O}_{Y,y}$ is a flat homomorphism where $x := f(y)$. We call f a *flat morphism of schemes* if it is flat at every point $y \in Y$.

Equivalently we say that the morphism of schemes $f : Y \to X$ is flat if for any point $y \in Y$ we have open affine neighborhoods $U = \operatorname{Spec} B$ of y and $V = \operatorname{Spec} A$ of $x := f(y)$ such that $f(U) \subseteq V$ and the induced map

$$f^\# : A \to B$$

makes B into a flat A-algebra.

Proposition 5.36. *Let $f : Y \to X$ be a flat morphism of schemes*

(1) (Composition) *If $g : Z \to Y$ is another flat morphisms of schemes, then the composition $f \circ g : Z \to X$ is flat.*

(2) (Base Change) If $f : Z \to X$ is another morphism, then the canonical map

$$p : Y \times_X Z \to Z$$

is flat

Proof. See Stacks Project [2017, Tag 01U7] for the composition and Stacks Project [2017, Tag 01U9] for the base change. □

Lemma 5.37. *A morphism of schemes that is locally of finite presentation and flat, is open.*

Proof. See Görtz and Wedhorn [2010, Thm. 14.33] □

Definition 5.38. We say that the morphism of schemes $f : Y \to X$ is *faithfully flat* if it is flat and surjective.

Definition 5.39. A morphism of schemes $f : Y \to X$ is called *affine* if the pre-image of any open affine subset of X under f is an open affine subset of Y

By definition it is clear that finite morphisms are affine.

The following lemma is an exercise in Hartshorne [1977]

Lemma 5.40. *Affine morphisms are seperated.*

We will need the following lemma when proving the existence of the étale fundamental group.

Definition 5.41. A morphism of schemes $f : Y \to X$ is said to be *étale* if it is flat and unramified. That is to say, f is étale if for every $y \in Y$ there exist open affine neighborhoods $U = \operatorname{Spec} B$ of y and $V = \operatorname{Spec} A$ of $x := f(y)$ such that $f(U) \subseteq V$ and the induced map

$$f^{\#} : A \to B$$

gives B the structure of an étale A-algebra.

Since being flat and being unramified are properties that are stable under composition and base change, see propositions 5.33 and 5.36, the same holds for the property of being étale.

Since étale morphisms are unramified they are locally of finite type and of course they are flat by definition. We then immediately get the following corollary to 5.37.

Corollary 5.42. *Let X be a scheme and $f : Y \to X$ be an étale morphism. Then f is open.*

Furthermore.

Lemma 5.43. *Any open immersion $Y \to X$ is étale.*

Proof. See [Stacks Project, 2017, Tag 02GP]. ☐

5.2. **The étale fundamental group.**

Definition 5.44. Let X be a scheme. A *finite étale cover* of X is a finite étale morphism

$$f : Y \to X$$

We denote by $\mathbf{F\acute{E}t}_X$ the full subcategory of $\mathbf{Sch/X}$ whose objects are finite étale covers of X.

Remark 5.45. By 5.8 we know that finite étale maps are closed and by 5.42 they are open as well. We thus see that finite étale morphisms are both open and closed, and so since we assume that our base scheme is connected we automatically get surjectivity from finite and étale, as long as we are not working with the empty cover. That is to say, a finite étale cover of X is either the empty cover or it is surjective.

Let X be a scheme. A *geometric point* in X is a morphism

$$\overline{x} : \operatorname{Spec} \Omega \to X$$

where Ω is an algebraically closed field.

If $f : Y \to X$ is a finite étale cover and $\overline{y} : \operatorname{Spec} \Omega_1 \to Y$ and $\overline{x} : \operatorname{Spec} \Omega_2 \to X$ are geometric points we say that $\overline{y}$ *lies over* $\overline{x}$ if there exists an algebraically closed field Ω that contains both Ω_1 and Ω_2 and such that the diagram

$$\begin{tikzcd} \operatorname{Spec}\Omega \arrow[r] \arrow[dr] & \operatorname{Spec}\Omega_1 \arrow[r, "\overline{y}"] & Y \arrow[d, "f"] \\ & \operatorname{Spec}\Omega_2 \arrow[r, "\overline{x}"] & X \end{tikzcd}$$

commutes.

Now if we are given a scheme X and we fix a geometric point $\overline{x} : \operatorname{Spec}\Omega \to X$ we obtain a functor

$$\mathbf{Fib}_{\overline{x}} : \mathbf{F\acute{E}t}_X \to \mathbf{FinSet}$$

given by

$$\mathbf{Fib}_{\overline{x}}(Y) = Y \times_X \operatorname{Spec}\Omega$$

i.e. $\mathbf{Fib}_{\overline{x}}$ is the fibre of the finite étale cover over the designated base point $\overline{x}$.

This lemma gives characterizations of finite flat morphisms and finite unramified morphisms.
We follow the proof in Cadoret [2013, Lem. 5.2] quite closely.

Lemma 5.46. *Let $f : Y \to X$ be a finite morphism. Then*

(1) *f is flat if and only if $f_*\mathcal{O}_Y$ is a locally free $\mathcal{O}_X$-module.*
(2) *The following properties are equivalent*
   (a) *$f : Y \to X$ is unramified*
   (b) *The diagonal morphism*

$$\Delta_{Y/X} : Y \to Y \times_X Y$$

   is an open immersion
   (c) *$(f_*\mathcal{O}_Y)_x \otimes_{\mathcal{O}_{X,x}} \kappa(x)$ is a finite étale algebra over $\kappa(x)$, $x \in X$*
   (d) *For any $y \in Y$ we have $\mathfrak{m}_{f(y)}\mathcal{O}_{Y,f(x)} = \mathfrak{m}_y$ and $\kappa(y)$ is a finite seperable extension of $\kappa(f(y))$.*

Proof. (1) We start by looking at the direction "$\Rightarrow$" Both the properties of flatness of $f : Y \to X$ and the local freeness of $f_*\mathcal{O}_Y$ as an $\mathcal{O}_X$-module are local. Therefore we can assume that $f : Y \to X$ is induced by a finite, flat A-algebra $f^\# : A \to B$ and furthermore since we assume we are working with locally Noetherian schemes, we may assume that A is Noetherian. Then we know that B is a flat A-module if and only if $B_\mathfrak{p}$ is a flat $A_\mathfrak{p}$-module for any prime ideal $\mathfrak{p} \subseteq A$. But $A_\mathfrak{p}$ is a

Noetherian local ring, so $B_{\mathfrak{p}}$ is a flat $A_{\mathfrak{p}}$-module if and only if it is free. We can therefore write

$$B_{\mathfrak{p}} = \bigoplus_{i=1}^{r} A_{\mathfrak{p}} \frac{b_i}{s}$$

for some $s \in A \backslash \mathfrak{p}$. This defines an exact sequence of $A_{\mathfrak{p}}$-modules

$$0 \to K \to A_s^r \xrightarrow{(\frac{b_1}{s},\dots\frac{b_r}{s})} B_s \to Q \to 0$$

Now A_s is Noetherian so the kernel K is a finitely generated A_s-module. Therefore the support

$$\mathrm{Supp}(K) := \{\mathfrak{q} \in A_s | K_{\mathfrak{q}} \neq 0\}$$

is closed and equal to

$$\mathrm{Supp}(K) = V(\mathrm{Ann(K)}) \subseteq \mathrm{Spec\,A_s}$$

Similarly since B_s is a finitely generated A_s-module and A_s is Noetherian, we have that the cokernel Q is a finitely generated A_s-module and therefore

$$\mathrm{Supp}(Q) = V(\mathrm{Ann(Q)}) \subseteq \mathrm{Spec\,A_s}$$

Now consider the ideal

$$I := \mathrm{Ann(K)} + \mathrm{Ann(Q)} \subseteq \mathrm{A_s}$$

We have

$$V(\mathrm{Ann(K)}) \cap \mathrm{V(Ann(Q))} = \mathrm{V(I)}$$

and $V(I)$ is isomorphic (as schemes) to $\mathrm{Spec}\,(A_s/I)$ i.e. it's an affine open subscheme of A_s. Now the localization map

$$A \to A_s$$

induces an isomorphism of schemes

$$\mathrm{Spec}\,(A_s) \to D(s)$$

and under this isomorphism the open subscheme $V(I)$ gets sent to an open subscheme of $D(f)$. Denote this open subscheme (when viewed as a subscheme of $\mathrm{Spec}\,A$) by $U_{\mathfrak{p}}$. Basically this consists of the points $\mathfrak{q}$ in $D(s) \subseteq \mathrm{Spec}\,A$ for which the map

$$A_{\mathfrak{q}}^r \xrightarrow{(\frac{b_1}{s},\dots,\frac{b_r}{s})} B_{\mathfrak{q}}$$

is an isomorphism. By construction $\mathfrak{p} \in U_{\mathfrak{p}}$ and

$$f_*\mathcal{O}_Y|_{U_{\mathfrak{p}}} \cong \mathcal{O}_X^r|_{U_{\mathfrak{p}}}$$

We see the direction"$\Leftarrow$" as follows. As in the proof of the other direction we notice that we can assume that

$$f : Y \to X$$

is induced by a free A-algebra of finite rank

$$f^{\#} : A \to B$$

Free modules are flat so B is a flat A-algebra. Therefore $B_{\mathfrak{p}}$ is a flat $A_{\mathfrak{p}}$-module for all prime ideals $\mathfrak{p} \subset A$. This shows that $f : \operatorname{Spec}(B) \to \operatorname{Spec}(A)$ is a flat morphism.

(2) We begin by proving $(a) \Rightarrow (b)$. So we assume that $f : Y \to X$ is finite and unramified. Since f is a finite morphism, it is affine and thus seperated (see 5.40). That means that the diagonal

$$\Delta_{Y/X} : Y \to Y \times_X Y$$

is a closed immersion. In particular, this means that

$$\Delta_{Y/X}(Y) = \operatorname{Supp}((\Delta_{Y/X})_* \mathcal{O}_Y)$$

Now let us look at the corresponding sheaf of ideals

$$\mathcal{I} := \ker(\Delta^{\#}_{Y/X} : \mathcal{O}_{Y \times_X Y} \to (\Delta_{Y/X})_* \mathcal{O}_Y) \subseteq \mathcal{O}_{Y \times_X Y}$$

By assumption we have

$$\Omega^1_{Y/X} = 0 = \Delta^*_{Y/X}(\mathcal{I}/\mathcal{I}^2)$$

and in particular

$$\mathcal{I}_{\Delta_{Y/X}(y)}/\mathcal{I}^2_{\Delta_{Y/X}(y)} = (\Delta^*_{Y/X}(\mathcal{I}/\mathcal{I}^2))_y = 0$$

or equivalently

$$\mathcal{I}_{\Delta_{Y/X}(y)} = \mathcal{I}^2_{\Delta_{Y/X}(y)}$$

for all $y \in Y$. Now $f : Y \to X$ is finite, hence affine and X is locally Noetherian so therefore Y is locally Noetherian and also the fibre product $Y \times_X Y$. Thus the category of coherent sheaves on $Y \times_X Y$ is Abelian and in particular the sheaf $\mathcal{I}$ is coherent (as it is a kernel of a map of coherent sheaves).

Now we can use Nakayamas lemma to conclude that

$$\mathcal{I}_{\Delta_{Y/X}(y)} = 0$$

for all $y \in Y$. Therefore $\Delta_{Y/X}(Y)$ is in the open subset

$$U := Y \times_X Y \setminus \operatorname{Supp}(\mathcal{I}) \subseteq Y \times_X Y$$

Now on the other hand, for any $u \in U$ the induced morphism on stalks

$$\Delta^{\#}_{Y/X,u} : \mathcal{O}_{Y \times_X Y,u} \xrightarrow{\cong} ((\Delta_{Y/X})_* \mathcal{O}_{Y,f(x)})_u$$

is an isomorphism. U is thus contained in $\operatorname{Supp}((\Delta_{Y/X})_*\mathcal{O}_Y) = \Delta_{Y/X}(Y)$. Hence

$$\Delta_{Y/X}(Y) = U$$

and the diagonal map

$$\Delta_{Y/X} : Y \to Y \times_X Y$$

is an open immersion.

Let us now prove $(b) \Rightarrow (c)$.

Fix an algebraically closed field Ω and consider two geometric points

$$\bar{x} : \operatorname{Spec}\Omega \to X$$

and

$$\bar{y} : \operatorname{Spec}\Omega \to Y_{\bar{x}}$$

where $Y_{\bar{x}}$ is the fibre product $Y \times_X \operatorname{Spec}\Omega$. We get a commutative diagram

$$\begin{tikzcd}
\operatorname{Spec}\Omega \arrow[r, "\bar{y}"] \arrow[d, "(id_{\operatorname{Spec}\Omega}{,}\bar{y})"'] & Y_{\bar{x}} \arrow[r] \arrow[d, "\Delta_{Y_{\bar{x}}/\operatorname{Spec}\Omega}"] & Y \arrow[d, "\Delta_{Y/X}"] \\
\operatorname{Spec}\Omega \times_{\operatorname{Spec}\Omega} Y_{\bar{x}} \arrow[r, "(\bar{y}{,}id_{Y_{\bar{x}}})"] & Y_{\bar{x}} \times_{\operatorname{Spec}\Omega} Y_{\bar{x}} \arrow[r] & Y \times_X Y
\end{tikzcd}$$

Open immersions are stable under base chance, so the geometric point $\bar{y} : \operatorname{Spec}\Omega \to Y_{\bar{x}}$ is an open immersion. It thus induces an isomorphism onto an open and closed subscheme of $Y_{\bar{x}}$. But notice that $Y_{\bar{x}}$ is finite (it's simply the fibre of the finite map $f : Y \to X$ over $\bar{x}$) and $\operatorname{Spec}\Omega$ is connected so it is an isomorphism from $\operatorname{Spec}\Omega$ to a connected component of $Y_{\bar{x}}$ and we thus get a decomposition

$$Y_{\bar{x}} = \coprod_{\bar{y}:\operatorname{Spec}\Omega \to Y_{\bar{x}}} \operatorname{Spec}\Omega$$

But then $(f_*\mathcal{O}_Y)_x \otimes_{\mathcal{O}_{X,x}} \kappa(x)$ is a product of $|Y_{\bar{x}}|$ copies (in particular a finite number of copies) of Ω which is an algebraically closed (in particular seperable) extension of $\kappa(x)$ so it's an étale $\kappa(x)$-algebra.

Let us now prove $(c) \Rightarrow (d)$.

We are looking here at local properties, so we may assume that the map $f : Y \to X$ is induced by a finite A-algebra $f^\# : A \to B$ where A is Noetherian. Consider a prime $\mathfrak{p} \subseteq A$, then our assumption is that as $\kappa(\mathfrak{p})$-algebras

$$B \otimes_A \kappa(\mathfrak{p}) = \prod_{1 \le i \le n} k_i$$

where each k_i is a seperable extension of $\kappa(\mathfrak{p})$. In particular, any prime ideal is maximal and equal to one of the following

$$\mathfrak{m}_j := \ker(\prod_{1\leq i\leq n}(k_i \twoheadrightarrow k_j))$$

with $1 \leq j \leq n$. But then if $\mathfrak{q}$ is in Y, lies above $\mathfrak{p} \in X$ and has image $\mathfrak{m}_j$ in $\operatorname{Spec}(B \otimes_A \kappa(\mathfrak{p}))$ for some j, and for that j we get

$$B_{\mathfrak{q}} \otimes_{A_{\mathfrak{p}}} \kappa(\mathfrak{p}) = (B \otimes_A \kappa(\mathfrak{p}))_{\mathfrak{m}_j} = k_j$$

which is by assumption a finite seperable field extension of $\kappa(\mathfrak{p})$.

Finally we prove $(d) \Rightarrow (a)$.

Now recall that a sheaf of Abelian groups on a topological space is zero if and only if its stalk is zero at every point. Thus the question of whether $\Omega^1_{Y/X}$ is zero becomes local and we may assume that the map $f : Y \to X$ is as before induced by a finite A-algebra

$$f^{\#} : A \to B$$

where A is Noetherian. Now $\Omega^1_{B/A}$ is a finitely generated B-module so Nakayama's lemma says that it is enough to show that

$$\Omega^1_{B/A} \otimes_B \kappa(\mathfrak{q}) = 0$$

for any $\mathfrak{q} \in Y$. But by assumption we have that for any $\mathfrak{q}$ lying above $\mathfrak{p} \in X$

$$B_{\mathfrak{q}} \otimes_{A_{\mathfrak{p}}} \kappa(\mathfrak{p}) = \kappa(\mathfrak{q})$$

and therefore

$$\begin{aligned}
\Omega^1_{B/A} \otimes_B \kappa(\mathfrak{q}) &= \Omega^1_{B/A} \otimes_A \kappa(\mathfrak{p}) \\
&= \Omega^1_{B\otimes_A \kappa(\mathfrak{p})/\kappa(\mathfrak{p})} \\
&= \Omega_{\kappa(\mathfrak{q})/\kappa(\mathfrak{p})} \\
&= 0
\end{aligned}$$

The last equality follows from the fact that $\kappa(\mathfrak{q})/\kappa(\mathfrak{p})$ is a finite seperable field extension, see Hartshorne [1977, Chap. II, thm. 8.6A.].

□

Definition 5.47. If $f : Y \to X$ is a finite étale cover then the size of the fiber of f over $\overline{x}$ is independent of the choice of a geometric point $\overline{x} : \operatorname{Spec} \Omega \to X$, and we call this size the *rank of* $f : Y \to X$ and denote it by $r(Y)$ or $r(f)$.

Lemma 5.48. *A finite étale cover $f : Y \to X$ is an isomorphism if and only if $r(f) = 1$.*

Proof. See Cadoret [2013, Cor. 5.9]. □

The main goal of this chapter is to proof that the category of finite étale covers of a fixed connected locally Noetherian base scheme is a Galois category and to introduce the étale fundamental group of this scheme. The following lemma is an important step in this proof, namely to show that categorical quotients by finite groups of automorphism exist in this category. In the proof of the lemma we follow Cadoret [2013, Lem. 5.12.(Step 1-1)] closely.

We first recall the definition of a totally split morphism of schemes.

Definition 5.49. A morphism $f : Y \to X$ is said to be totally split if X can we written as a disjoint union $X = \coprod X_i$ such that for each i, $f^{-1}(X_i)$ can be written as a disjoint uniont of n copies of X_i.

We notice that under the hypothesis of this subsection, X is connected and thus a morphism $f : Y \to X$ is totally split if Y is isomorphic to the disjoint union of some finite number of copies of X.

Clearly totally split morphisms are finite and étale.

Lemma 5.50. *An affine, surjective morphism $f : Y \to X$ is a finite étale cover of X if and only if there exists a finite faithfully flat morphism $g : X^{'} \to X$ such that the first projection*

$$Y^{'} := X^{'} \times_X Y \xrightarrow{g^{'}} X^{'}$$

is a totally split finite étale cover of $X^{'}$

Proof. We prove one direction at a time.

"⇒" Since $g : X^{'} \to X$ is finite and faithfully flat we know from part 1 of 5.46 that for any $x \in X$ there exists an open affine neighborhood $U = \mathrm{Spec}\,(A)$ of x such that the restriction

$$g|^{U}_{g^{-1}(U)} : g^{-1}(U) \to U$$

is induced by a finite A-algebra

$$g^{\#} : A \hookrightarrow A^{r_1}$$

for some integer $r_1 \geq 1$. Now since $f : Y \to X$ is affine and surjective (by assumption) the restricted map

$$f|^{U}_{f^{-1}(U)} : f^{-1}(U) \to U$$

corresponds to an A-algebra

$$f^{\#} : A \hookrightarrow B$$

By assumption we have $B \otimes_A A^{'} = A^{'r_2}$ as $A^{'}$-algebras, for some integer $r_2 \geq 1$. Then as A-modules we have

$$B \otimes_A A^{'} = A^{r_1 r_2}$$

On the other hand we also have the equality of B-modules, and hence of A-modules,

$$B \otimes_A A^{'} = B^{r_1}$$

This means in particular that B is a direct summand of $A^{r_1 r_2}$ as A-modules so it is projective and thus flat. This shows the flatness of $f : Y \to X$. Now since A is Noetherian and B is a submodule of the finitely generated A-module $A^{r_1 r_2}$ we find that B is a finitely generated A-module. Hence $f : Y \to X$ is a finite morphism.

What now remains is to show that $f : Y \to X$ is unramified. Denote the projections in the following pullback diagram by $f^{'}$ and $g^{'}$

$$\begin{tikzcd} Y^{'} \arrow[r, "g^{'}"] \arrow[d, "f^{'}"'] & Y \arrow[d, "f"] \\ X \arrow[r, "g"] & X \end{tikzcd}$$

But by assumption $f^{'}$ is étale so in particular it is unramified. Thus $\Omega^1_{Y^{'}/X^{'}} = 0$. Now by basic properties of Kähler differentials 5.27, we have

$$(g^{'})^* \Omega^1_{Y/X} = \Omega^1_{Y^{'}/X^{'}} = 0$$

i.e.

$$((g^{'})^* \Omega^1_{Y/X})_{y^{'}} = \Omega^1_{Y^{'}/X^{'}, g(y^{'})} = 0$$

for all $y^{'} \in Y^{'}$. Since by assumption $g^{'} : Y^{'} \to Y$ is the base change of a surjective map $g : X^{'} \to X$ it is surjective, and hence the stalk of $\Omega^1_{Y/X}$ is zero at every point $y \in Y$.

"$\Leftarrow$" We prove this by induction on the rank of the étale cover $f : Y \to X$. If $r(f) = 1$, then we know that $f : Y \to X$ is an isomorphism and the statement becomes obvious, we can take $g = id_X$. So we assume $r(f) > 1$. Part 2-(b) of 5.46 then tells us that since f is a finite étale cover, it is finite and unramified, so the diagonal map

$$\Delta_{Y/X} : Y \to Y \times_X Y$$

is both an open and a closed immersion. Therefore we can decompose the fibre product as

$$Y \times_X Y = Y \amalg Y^{'}$$

where we have made the identification of Y with it's image $\Delta_{Y/X}$ in $Y \times_X Y$ and $Y^{'}$ is simply the complement $Y^{'} = Y \times_X Y \backslash \Delta_{Y/X}$. In particular, the inclusion

$$\iota : Y^{'} \hookrightarrow Y \times_X Y$$

is both an open immersion and a closed immersion and so is a finite étale morphism by 5.9 and 5.43. But by assumption $f : Y \to X$ is finite étale and thus the base change

$$Y \times_X Y \xrightarrow{p_1} Y$$

is finite étale. The composition of these two

$$f^{'} : Y^{'} \xrightarrow{\iota} Y \times_X Y \xrightarrow{p_1} Y$$

is finite and étale. But

$$p_1 \circ \Delta_{Y/X} = id_X$$

i.e. $\Delta_{Y/X}$ is a section of p_1 so we get

$$r(f^{'}) = r(p_1) - 1 = r(f) - 1$$

By the induction hypothesis we then have a finite flat morphism $g : X^{'} \to Y$ such that the base change map π_1

$$\begin{tikzcd} X^{'} \times_Y Y^{'} \arrow[r] \arrow[d, "\pi_1"'] & Y^{'} \arrow[d, "f^{'}"] \\ X^{'} \arrow[r, "g"'] & Y \end{tikzcd}$$

is a totally split étale cover of $X^{'}$. But then the composite map

$$f \circ g : X^{'} \to X$$

is also finite and faithfully flat. What remains is to show that the map $X^{'} \times_X Y \to X^{'}$ is a totally split étale cover. Notice that the following diagrams commute:

$$\begin{tikzcd} X^{'} \times_Y (Y \times_X Y) \arrow[r, "g \circ q_1"] \arrow[d, "q_1"'] & Y \arrow[d, "f"] \\ X^{'} \arrow[r, "f \circ g"'] & X \end{tikzcd}$$

and

$$\begin{tikzcd} X^{'} \times_X Y \arrow[r, "(g,id_Y)"] \arrow[d] & Y \times_X Y \arrow[d, "p_1"] \\ X^{'} \arrow[r, "g"'] & Y \end{tikzcd}$$

where $q_1 : X^{'} \times_Y (Y \times_X Y) \to X^{'}$ is the canonical projection. This gives us unique maps $u_1 : X^{'} \times_Y (Y \times_X Y) \to X^{'} \times_X Y$ and $u_2 : X^{'} \times_X Y \to X^{'} \times_Y (Y \times_X Y)$ that are inverses of each other. That is, we have

$$\begin{aligned} X^{'} \times_X Y &\cong X^{'} \times_Y (Y \times_X Y) \\ &\cong X^{'} \times_Y (Y \amalg Y^{'}) \\ &\cong (X^{'} \times_Y Y) \amalg (X^{'} \times_Y Y^{'}) \end{aligned}$$

and $X^{'} \times_Y Y \cong Y$ and $X^{'} \times_Y Y^{'} \to X'$ is a totally split étale cover.

$\square$

We now look at a theorem due to Grothendieck. It tells us that faithfully flat morphisms of finite type in **Sch/X** are strict epimorphisms.

Theorem 5.51. *In the category **Sch/X**, faithfully flat morphisms of finite type are strict epimorphisms.*

Proof. See for example Milne [1980, Thm. 2.17] $\square$

Lemma 5.52. *Let $f : Y \to X$ and $g : Z \to X$ be finite étale morphisms such that the following diagram commutes*

$$\begin{tikzcd} Y \arrow[r, "u"] \arrow[d, "f"'] & Z \arrow[dl, "g"] \\ X & \end{tikzcd}$$

then the morphism $u : Y \to Z$ is finite étale as well.

Proof. Consider the graph of u (as morphism over X) identified with the base change

$$\begin{tikzcd} Y \arrow[r] \arrow[d, "\Gamma_u"'] & Z \arrow[d, "\Delta_{Z/X}"] \\ Y \times_X Z \arrow[r, "(u,id_Z)"'] & Z \times_X Z \end{tikzcd}$$

and consider the base chance map

$$\begin{tikzcd} Y \times_X Z \arrow[r] \arrow[d, "p_2"'] & Y \arrow[d, "f"] \\ Y \arrow[r, "g"'] & X \end{tikzcd}$$

Then we can write u as the composition $u = p_2 \circ \Gamma_u$. The diagonal morphism $\Delta_{Y/X}$ is finite and étale and hence the base change $\Gamma_u : Y \to Y \times_Z Y$ is finite étale. In the same manner we see that p_2 is finite étale and the composition u is then finite étale. $\square$

We now state and prove the main theorem of this chapter.

Theorem 5.53. *For any given scheme X with a fixed geometric point $\overline{x} : \operatorname{Spec} \Omega \to X$, $\textbf{FÉt}_X$ is a Galois category with a fibre functur $\textbf{Fib}_{\overline{x}}$.*

Proof. We verify the axioms one by one.

(1) (Final object and fibre products) The identity morphism $X \to X$ belongs to $\mathbf{FÉt}_X$ and it is clearly the final object in $\mathbf{FÉt}_X$. Now if $Y_1 \to Z \leftarrow Y_2$ are maps of étale covers of X, then stability under base chance and composition tells us that the canonical map f:

$$\begin{tikzcd} Y_1 \times_Z Y_2 \arrow[r, "p_2"] \arrow[d, "p_1"'] \arrow[dr, "f"] & Y_2 \arrow[d] \\ Y_2 \arrow[r] & Z \end{tikzcd}$$

is a finite étale map. Thus the composition $Y_1 \times_Z Y_2 \to Z \to X$ is then a finite étale map, so $\mathbf{FÉt}_X$ is closed under fibre products.

(2) Consider two finite étale covers $f_1 : Y_1 \to X$ and $f_2 : Y_2 \to X$. The coproduct exists in the category of schemes (it's the disjoint union) and we have the commutative diagram

$$\begin{tikzcd} & X & \\ Y_1 \arrow[ur, "f_1"] \arrow[r, "\iota_1"] & Y_1 \amalg Y_2 \arrow[u, dotted, "\exists! u"] & Y_2 \arrow[ul, "f_2"'] \arrow[l, "\iota_2"'] \end{tikzcd}$$

We want to conclude that $u : Y_1 \amalg Y_2 \to X$ is a finite étale cover. But the property of being finite étale is local so we may assume our maps are induced by maps of algebras $A \to B_1$ and $A \to B_2$ both of which are finite étale algebras. But then the canonical map $A \to B_1 \otimes B_2$ is clearly finite and étale.

A more involved question regards the existence of the quotient by finite groups of isomorphisms. We assume first that the base scheme is affine, $X = \operatorname{Spec} A$ (where A is a Noetherian ring). In this case the étale property (more precisely the finiteness) of $f : Y \to X$ allows us to assume that f is induced by a finite A-algebra $f^{\#} : A \to B$. The equivalence of categories

$$Aff/_{\operatorname{Spec} A} \cong A - \mathrm{Alg}$$

where $Aff/_{\operatorname{Spec} A}$ is the category of affine $\operatorname{Spec} A$-schemes, tells us that the factorisation

$$\begin{tikzcd} Y \arrow[r, "p_G"] \arrow[d, "f"'] & \operatorname{Spec}(B^{G^{op}}) =: G \setminus X \arrow[dl, "f_G"] \\ X & \end{tikzcd}$$

is the categorical quotient of $f : Y \to \operatorname{Spec} A$ in $Aff/_{\operatorname{Spec} A}$.

What we have left to show is that the map $f_G : G \setminus Y \to X$ is a finite étale cover.

The map $f : Y \to X$ is a finite étale cover, so 5.50 tells us that there exists a faithfully flat A-algebra $A \to A^{'}$ such that

$$B \otimes_A A^{'} \cong A^{'n}$$

We have an exact sequence of A-algebras

$$0 \to B^{G^{op}} \hookrightarrow B \xrightarrow{\psi := \sum_{g \in G^{op}} (id_B - g)} \bigoplus_{g \in G^{op}} B$$

that we can tensor by the flat A-algebra $A^{'}$ to obtain an exact sequence of $A^{'}$-algebras

$$0 \to B^{G^{op}} \otimes_A A^{'} \hookrightarrow B \otimes_A A^{'} \xrightarrow{\psi \otimes id_{A^{'}}} \bigoplus_{g \in G^{op}} B \otimes_A A^{'}$$

Therefore

$$B^{G^{op}} \otimes_A A^{'} \cong \ker(\psi \otimes id_{A^{'}})$$

i.e.

$$B^{G^{op}} \otimes_A A^{'} \cong (B \otimes_A A^{'})^{G^{op}} \cong (A^{'n})^{G^{op}}$$

Notice that G^{op} is a subgroup of the automorphism group of the free $A^{'}$-algebra $A^{'n}$ which is of course the symmetric group, $\mathcal{S}_n$, on n-letters. We can therfore realize it as acting on the integer interval $[n]$ and we have

$$(A^{'n})^{G^{op}} = \bigoplus_{G^{op} \setminus [n]} A^{'}$$

Scheme-theoretically we have that if $\phi : X^{'} \to X$ is the faithfully flat morphism corresponding to $A \to A^{'}$, then $X^{'} \times_X Y$ is a coproduct of n copies of $X^{'}$ which G acts on by permutation and we have the isomorphisms

$$X^{'} \times_X (G \setminus Y) \cong G \setminus (\coprod_{[n]} X^{'}) \cong \coprod_{G \setminus [n]} X^{'}$$

By 5.50, we have

$$f_G : G \setminus Y \to X$$

is a finite étale cover in the case where X is affine.

Now let us look at the general case where X is a connected locally Noetherian scheme. We can cover X by open affine sub-schemes $\{X_i = \operatorname{Spec} A_i\}_{i \in I}$ such that for each i the map restriction

$$f_i := f_{f^{-1}(X_1)} : f^{-1}(X_i) \to X_i$$

is induced by a finite A_i-algebra

$$f_i^{\#} : A_i \to B_i$$

Let $G \setminus Y$ denote the topological space obtained by quotienting the underlying topological space of Y by the action of G. Then we endow the space with the structure of a scheme by pulling back the structure sheaf $\mathcal{O}_X$ locally by the quotient maps $f_{i,G} : \operatorname{Spec}(B_i^{G^{op}}) \to X$. This makes the diagram

$$\begin{tikzcd} Y \arrow[r, "p_G"] \arrow[d, "f"'] & G \setminus Y \arrow[dl, "f_G"] \\ X & \end{tikzcd}$$

commutative as a diagram of schemes and maps of schemes. It is clear by construction that $G \setminus Y$ satisfies the universal property of categorical quotients, and that

$$f_G : G \setminus Y \to X$$

is a finite étale cover.

(3) Let $f_1 : Y_1 \to X$ and $f_2 : Y_2 \to X$ be two finite étale covers and $u : Y_1 \to Y_2$ be a map in $\mathbf{F\acute{E}t}_X$. By 5.52 we know that u is finite étale and therefore both open (étale morphisms are open by 5.42) and closed (finite morphisms are closed by 5.8). Therefore we can factor Y as the coproduct

$$Y_2 = Y_2^{'} \amalg Y_2^{''}$$

where $Y_2' := u(Y_1)$ and $Y_2'' := Y_2 \setminus Y_2'$ are both open and closed. The morphism u then factors as

$$\begin{tikzcd} Y_1 \arrow[r, "u"] \arrow[rr, bend right, "u"'] & Y_2' \arrow[r, "\iota"] & Y_2 = Y_2' \amalg Y_2'' \end{tikzcd}$$

where $\iota : Y_2' \hookrightarrow Y_2$ is the inclusion. Since $u : Y_1 \to Y_2'$ is a surjection, it is a faithfully flat morphism and thus by 5.51 it is a strict epimorphism. The inclusion $\iota : Y_2' \hookrightarrow Y_2$ is an open immersion and thus a monomorphism.

(4) By the definition of the rank of an étale cover we see that $\mathbf{Fib}_{\overline{x}}(f : Y \to X)$ is a single point if and only if $r(f) = 1$. But as we showed this happens if and only if $f : Y \xrightarrow{\cong} X$ is an isomorphism. So $\mathbf{Fib}_{\overline{x}}$ sends final objects of $\mathbf{F\acute{E}t}_X$ to final objects of $\mathbf{FinSet}$.

Let us now look at the fibre product of two objects of $\mathbf{F\acute{E}t}_X$

$$\begin{tikzcd} Y_1 \times_X Y_2 \arrow[r, "p_1"] \arrow[d, "p_2"'] \arrow[dr, dashed, "f"] & Y_1 \arrow[d, "f_1"] \\ Y_2 \arrow[r, "f_2"'] & X \end{tikzcd}$$

We want to show that $\mathbf{Fib}_{\overline{x}}$ commutes with this fibre product, i.e. that

$$\mathbf{Fib}_{\overline{x}}(f : Y_1 \times_X Y_2 \to X) = \mathbf{Fib}_{\overline{x}}(Y_1) \times \mathbf{Fib}_{\overline{x}}(Y_2)$$

Notice that in the right hand side we are taking the normal cartesian product since we are taking the fibre product over the single point $\mathbf{Fib}_{\overline{x}}(id_X)$. When we view $\mathbf{Fib}_{\overline{x}}$ as the functor $Y \to Y \times_X \operatorname{Spec}\Omega$, basic properties of the fibre product give us

$$\begin{aligned} \mathbf{Fib}_{\overline{x}}(Y_1 \times_X Y_2) &= (Y_1 \times_X Y_2) \times_X \operatorname{Spec}\Omega \\ &\cong (Y_1 \times_X Y_2) \times_X (\operatorname{Spec}\Omega \times_{\operatorname{Spec}\Omega} \operatorname{Spec}\Omega) \\ &\cong (Y_1 \times_X \operatorname{Spec}\Omega) \times_{\operatorname{Spec}\Omega} (Y_2 \times_X \operatorname{Spec}\Omega) \\ &= \mathbf{Fib}_{\overline{x}}(Y_1) \times \mathbf{Fib}_{\overline{x}}(Y_2) \end{aligned}$$

(5) In the category of finite sets the strict epimorphisms are simply the epimorphisms, i.e. the surjective maps, so it is clear that $\mathbf{Fib}_{\overline{x}}$ sends strict epimorphisms to strict epimorphisms. Viewing $\mathbf{Fib}_{\overline{x}}$ as the functor

$$\mathbf{Fib}_{\overline{x}}(Y) = Y \times_X \operatorname{Spec}\Omega$$

we immediately see that if $Y = Y_1 \amalg Y_2$ in $\mathbf{F\acute{E}t}_X$ then

$$\begin{aligned}\mathbf{Fib}_{\overline{x}}(Y) &= (Y_1 \amalg Y_2) \times_X \operatorname{Spec} \Omega \\ &= (Y_1 \times_X \operatorname{Spec} \Omega) \amalg (Y_2 \times_X \operatorname{Spec} \Omega) \\ &= \mathbf{Fib}_{\overline{x}}(Y_1) \amalg \mathbf{Fib}_{\overline{x}}(Y_2)\end{aligned}$$

so the fibre functor commutes with finite coproducts.

Now let $f : Y \to X$ be a finite étale cover and G be a finite group acting by automorphism on it. Whether the fibre functor $\mathbf{Fib}_{\overline{x}}$ commutes with passing to the quotient is local in the sense that it is equivalent to the same question after a base change by a finite faithfully flat morphism $X' \to X$ and thus by 5.50 we may assume that $f : Y \to X$ was totally split to begin with (and the group G acts by permutations of the isomorphic copies of X). But then this is trivial since now we have

$$G \setminus Y = \coprod_{G \setminus \mathbf{Fib}_{\overline{x}}(Y)} X$$

(6) One direction here is clear, namely that an isomorphism of finite étale covers induces a bijection on the fibres.

For the other direction we let $f : Y \to X$ and $g : Z \to X$ be two finite étale covers in $\mathbf{F\acute{E}t}_X$, and let $\phi : Y \to Z$ be a morphism of X-schemes. Assume that

$$\mathbf{Fib}_{\overline{x}}(\phi) : \mathbf{Fib}_{\overline{x}}(Z) \to \mathbf{Fib}_{\overline{x}}(Y)$$

is a bijection. Then by 5.52 the map ϕ is finite and étale. Furthermore, it is surjective by assumption. Hence $\phi : Y \to Z$ is a finite étale cover and it has rank 1 since $\mathbf{Fib}_{\overline{x}}(\phi)$ is bijective. Hence by 5.48 ϕ is an isomorphism as we wanted.

□

Definition 5.54. The étale fundamental group of a scheme X with a base point $\overline{x}$ is defined as the fundamental group of the Galois category $(\mathbf{F\acute{E}t}_X, \mathbf{Fib}_{\overline{x}})$. We denote it by $\pi_1^{\text{ét}}(X, \overline{x})$.

The fundamental theorem of Galois categories then immediately gives us the following theorem.

Theorem 5.55. *Let X be a connected and locally Notherian scheme and $\overline{x}$ be a geometric point in X. Then the fibre functor $\boldsymbol{Fib}_{\overline{x}}$ induces an equivalence of categories*

$$\boldsymbol{F\acute{E}t}_X \simeq \pi_1^{\text{ét}}(X, \overline{x}) - \mathrm{Rep}_{\mathrm{c}}$$

We also get immediately that it does not matter what geometric point we choose to construct our fundamental group.

Proposition 5.56. *Let as before X be a locally Noetherian connected scheme. Let $\overline{x}_1$ and $\overline{x}_2$ be two geometric points in X. Then there is a non-canonical continuous isomorphism*

$$\pi_1^{\text{ét}}(X, \overline{x}_1) \cong \pi_1^{\text{ét}}(X, \overline{x}_2)$$

Proof. This is a direct corollary of 4.25. □

5.3. **Examples.**

Example 5.57. Let k be a field. Choosing a geometric point $\bar{k} : \operatorname{Spec} \Omega \to \operatorname{Spec} k$ amounts to fixing an algebraic closure Ω of k. Now a finite étale cover

$$Y \to \operatorname{Spec} k$$

is finite, and in particular affine, so it is equivalent to a finite étale k-algebra, i.e

$$Y = \operatorname{Spec} A$$

where

$$A = \prod_{i=1}^{n} L_i$$

where each L_i/k is a seperable extension contained in Ω/k. Thus if k_s is the seperable closure of k in Ω (which is of course isomorphic as fields to any other seperable closure) we obtain

$$\pi_1^{\text{ét}}(\operatorname{Spec} k, \bar{k}) \cong \operatorname{Gal}(k_s/k)$$

i.e. the étale fundamental group of the point $\operatorname{Spec} k$ is isomorphic to the absolute Galois group of k.

Let us recall a definition.

Definition 5.58. Let X be a scheme. We say that X is *normal* if the local ring $\mathcal{O}_{X,x}$ is an integral domain, and integrally closed in it's field of fractions.

We need the following lemma for our next example.

Lemma 5.59. *Any connected normal and Noetherian scheme is irreducible.*

Proof. Any connected normal and Noetherian scheme is integral, see Stacks Project [2017, Tag 033M] and integral schemes are irreducible, see Stacks Project [2017, 010N]. □

Example 5.60. Let k be an algebraically closed field and consider the projective line $\mathbb{P}^1 := \mathbb{P}^1_k$ over k. It is an integral scheme and has a unique generic point η, and moreover it is normal. Consider the function field $k(T)$ of $\mathbb{P}^1$ and the canonical inclusion

$$\operatorname{Spec}(k(T)) \hookrightarrow \mathbb{P}^1$$

Now let $f : Y \to \mathbb{P}^1$ be connected finite étale cover. Then we know that the base change

$$\begin{tikzcd} \operatorname{Spec}(k(T)) \times_{\mathbb{P}^1} Y \arrow[r] \arrow[d] & Y \arrow[d] \\ \operatorname{Spec}((T)) \arrow[r] & \mathbb{P}^1 \end{tikzcd}$$

is normal. It is also Noetherian and so by the above lemma $\operatorname{Spec}(k(T)) \times_{\mathbb{P}^1} Y$ is connected. Therefore we have a connected étale cover

$$\operatorname{Spec}(k(T)) \times_{\mathbb{P}^1} Y \to \operatorname{Spec}(k(T))$$

and thus it is a point, and has rank 1. Therefore $Y \to \mathbb{P}^1$ has rank 1, and by 5.59 is an isomorphism.

Hence the only connected étale cover of $\mathbb{P}^1$ is the trivial one, from which we see that

$$\pi_1^{\text{ét}}(\mathbb{P}^1, \eta) = 0$$

Example 5.61. Let k be an algebraically closed field of characteristic 0, and denote by $\mathbb{G}_m := \mathbb{G}_m(k) := \mathbb{A}^1_k \setminus \{0\}$. Let $\overline{x} : \operatorname{Spec} k \to \mathbb{G}_m$ be any geometric point in $\mathbb{G}_m$. We can define a finite étale cover of $\mathbb{G}_m$ by

$$\begin{aligned} \phi_n : \mathbb{G}_m &\to \mathbb{G}_m \\ t &\mapsto t^n \end{aligned}$$

It is clearly Galois of rank n. The automorphism group of this cover, denote it by $\operatorname{Aut}_n(\mathbb{G}_m)$, is the group of all n-th roots of unity in k and since k is algebraically closed

$$\operatorname{Aut}_n(\mathbb{G}_m) \cong \mathbb{Z}/n\mathbb{Z}$$

Furthermore,these are all the Galois covers and therefore

$$\pi_1^{\text{ét}}(\mathbb{G}_m, \overline{x}) = \varprojlim_{n \in \mathbb{N}_+} \mathbb{Z}/n\mathbb{Z} = \widehat{\mathbb{Z}}$$

Example 5.62. Let k be an algebraically closed field. Let X be the nodal curve obtained by taking the projective line $\mathbb{P}^1 := \mathbb{P}^1_k$ over k and gluing together 0 and ∞ transversally. Let x be the node and $\overline{x}$ the corresponding geometric point $\overline{x} : \operatorname{Spec} k \to X$. We have a canonical map $\mathbb{P}^1 \to X$. Let $Y \to X$ be a connected étale cover of rank n and consider the base change

$$\begin{tikzcd} Y_{\mathbb{P}^1} := \mathbb{P}^1 \times_X Y \arrow[r] \arrow[d] & Y \arrow[d] \\ \mathbb{P}^1 \arrow[r] & X \end{tikzcd}$$

Now $Y_{\mathbb{P}^1} \to \mathbb{P}^1$ is an étale cover so if it has rank n then by the above example it decomposes as a cover of $\mathbb{P}^1$ into n copies of the trivial cover.

$$Y_{\mathbb{P}^1} = \coprod \mathbb{P}^1$$

Now if we remove x from X then we obtain an open subscheme $U \subset X$ and an étale cover $Y^{'} \to U$ where $Y^{'}$ is obtained from Y by removing the fibre over x. Notice that by construction the fibre of $\mathbb{P}^1 \to X$ over x is $\{0, \infty\}$ so by removing the fibre we obtain $\mathbb{G}_m$ We get a commutative diagram in the category of schemes

$$\begin{tikzcd} \coprod \mathbb{G}_m \arrow[r] \arrow[d] & Y^{'} \arrow[d] \\ \mathbb{G}_m \arrow[r] & U \end{tikzcd}$$

But notice that $\mathbb{G}_m \to U$ is an isomorphism so we obtain an isomorphism

$$\coprod \mathbb{G}_m \xrightarrow{\cong} Y^{'}$$

Now going back to the original diagram, we see that if $\mathbb{P}^1_i$ denotes the i-th cofactor in $Y_{\mathbb{P}^1}$ then $0 \in \mathbb{P}^1_i$ is mapped to 0 in $\mathbb{P}^1$ and thus to the node $x \in X$. Therefore it maps to some element in the fibre of $Y \to X$ over x. In this way we obtain a 2 to 1 mapping of the fibre in $Y_{\mathbb{P}^1}$, which is simply the 0's and ∞'s in each of the $\mathbb{P}^1_i$, to the fibre in Y. Now if $y \in Y$ is in the fibre over x and z_1, z_2 are the points of $Y_{\mathbb{P}^1}$ that map to y, then z_1 and z_2 can not come from the same cofactor $\mathbb{P}^1_i$ because that would induce an isomorphism $\mathbb{P}^1 \amalg \widehat{Y} \xrightarrow{\cong} Y$ where $\widehat{Y}$ is the image of $Y_{\mathbb{P}^1} \setminus \mathbb{P}^1_i$. In particular Y would not be connected, contradicting our assumption. This implies that up to the order of the $\mathbb{P}^1_i$'s in $Y_{\mathbb{P}^1}$, and automorphisms of the $\mathbb{P}^1_i$'s that switch between 0 and ∞ we can say that the 0 of $\mathbb{P}^1_2$ maps to the same point in Y as the ∞ of $\mathbb{P}^1_1$, the 0 of $\mathbb{P}^1_3$ maps to the same point in Y as the ∞ in $\mathbb{P}^1_2$ etc. until we reach that the 0 in $\mathbb{P}^1_1$ maps to the same point in Y as the ∞ in $\mathbb{P}^1_n$. In this way we see that Y is isomorphic to a chain of n copies of $\mathbb{P}^1$ glued together cyclically. We call such a connected étale cover of rank n, Y_n.

Now the automorphism group $\mathrm{Aut}_X(Y_n)$ is clearly $\mathbb{Z}/n\mathbb{Z}$ and we thus obtain

$$\pi_1^{\text{ét}}(X, \overline{x}) \cong \varprojlim \mathrm{Aut}_X(Y_n) \cong \widehat{\mathbb{Z}}$$

5.4. **Some interesting properties of the fundamental group.** Here we discuss some properties of the étale fundamental group. We shall not present proofs.

The first result we want to present relates the étale fundamental group of a scheme defined over a field k to the étale fundamental group of the pullback by $\mathrm{Spec}\,(k_s) \to \mathrm{Spec}\,(k)$ and the absolute Galois group of k, where k_s is some fixed seperable closure of k.

First we recall a definition.

Definition 5.63. A scheme X over a field k is called *geometrically integral* if $X \times_{\mathrm{Spec}\,(k)} \mathrm{Spec}\,(\bar{k})$ is integral for an algebraic closure $\bar{k}$ of k.

If a scheme is geometrically integral, then $X \times_{\mathrm{Spec}\,(k)} \mathrm{Spec}\,(L)$ is an integral scheme for any field extension L/k. In particular X is an integral scheme.

The following is Szamuely [2009, Prop. 5.6.1]

Proposition 5.64. *Let X be a quasi-compact and geometrically integral scheme over a field k. Fix an algebraic closure $\bar{k}$ of k and let k_s be the seperable closure of k in $\bar{k}$. Write*

$$\bar{X} := X \times_{\mathrm{Spec}\,(k)} \mathrm{Spec}\,(k_s)$$

and let $\overline{x} : \mathrm{Spec}\,(\bar{k}) \to \bar{X}$ be a geometric point. Then the sequence of profinite groups

$$1 \to \pi_1^{\text{ét}}(\bar{X}, \overline{x}) \to \pi_1^{\text{ét}}(X, \overline{x}) \to \mathrm{Gal}(k) \to 1$$

induced by the projection $\bar{X} \to X$ and the structure map $X \to \mathrm{Spec}\,(k)$ is exact.

Remark 5.65. Integral schemes are connected, so both the fundamental groups in 5.64 are well defined.

We also have a GAGA theorem. Recall from Serre [1956] that given a scheme X of finite type over $\mathbb{C}$, then we can associate with it a complex-analytic space X^{an} and with any sheaf $\mathcal{F}$ on X we can associate an analytic sheaf $\mathcal{F}^{\text{an}}$ on X^{an} in such a way that the functor $\mathcal{F} \to \mathcal{F}^{\text{an}}$ is exact.

We have the following Szamuely [2009, Thm. 5.7.4]

Proposition 5.66. *Let X be a connected scheme of finite type over $\mathbb{C}$. The functor $X \mapsto X^{an}$ induces an equivalence of the category of finite* *étale covers of X with that of finite topological covers of X^{an}. Therefore we have for any $\mathbb{C}$-point $\overline{x} : \mathrm{Spec}\,(\mathbb{C}) \to X$ an induced isomorphism*

$$\widehat{\pi_1^{top}(X^{an}, \overline{x})} \xrightarrow{\sim} \pi_1^{\text{ét}}(X, \overline{x})$$

This follws directly from the Riemann existence theorem, see Hartshorne [1977, App. B, Thm. 3.2] and the discussion that follows it.

6. Grothendieck topologies and l-adic sheaves

In general we follow Milne [2013]and Kindler and Rülling [2014] and recommend them for a fuller account.

6.1. Grothendieck topologies, sites, and sheaves.

Definition 6.1. Let $\mathcal{C}$ be a category that is closed under finite fibre products. A *Grothendieck topology* (or just *topology*) on $\mathcal{C}$ is an assignment for each object $X \in \mathcal{C}$ of coverings, i.e. collections of morphisms $\{U_i \to X\}$ in C such that the following properties hold:

1. (T1) If $\{U_i \to X\}$ is a covering and $Y \to X$ is a morphism in $\mathcal{C}$, then all fibre products $U_i \times_X Y$ exist and the family $\{U_i \times_X Y \to Y\}$ is a covering.
2. (T2) If $\{U_i \to X\}$ is a covering and for each i we have a covering $\{V_{ij} \to U_i\}$, then the collection of composite maps $\{V_{ij} \to X\}$ is a covering.
3. (T3) Any isomorphism in $\mathcal{C}$ is a covering, i.e. if $\phi : Y \to X$ is an isomorphism then the set $\{Y \xrightarrow{\phi} X\}$ is a covering.

If $\mathcal{C}$ is a category on which we have defined a Grothendieck topology, we denote the collection of coverings by $\mathcal{T}_{\mathcal{C}}$ or simply $\mathcal{T}$. We call the pair $(\mathcal{C}, \mathcal{T}_{\mathcal{C}})$ a *site*. The category $\mathcal{C}$ is then called the *underlying category* of the site. Notice that different sites can have the same underlying category, i.e. in general there can be more than one topology defined on a given category.

Now we give some examples of sites. The first and most basic example is the one from which the general definition is abstracted.

Example 6.2 (The site associated with a topological space)**.** Let X be a topological space. Then denote by $op(X)$ the category whose objects are the open sets $U \subset X$ and where there is a unique morphism for each inclusion $U \subset V$ (and no other). Let $\mathcal{T}_X$ be the collection of all families $\{U_i \to U\}$ for which $\cup_i U_i = U$, i.e. coverings in the "usual sense". Then $\mathcal{T}_X$ forms a topology on $op(X)$. Now let $f : Y \to X$ be a continuous map of topological spaces. Then we obtain a "pullback" morphism of sites: $F : (op(X), \mathcal{T}_X) \to (op(Y), \mathcal{T}_Y)$ where $F(U) = f^{-1}(U)$.

A special case of the previous example is important in algebraic geometry.

Example 6.3 (The Zariski site)**.** Let X be a scheme. Then in particular it is a topological space in the Zariski topology. We can then construct the site associated with this topological space, denoted X_{Zar}. We call this the *Zariski site on X*.

Definition 6.4. A morphism $f : Y \to X$ is said to be *fppf* (stands for fidèlement plat de présentation finie) if it is flat and locally finitely presented.

Example 6.5 (fppf site)**.** Let X be a scheme. The (large) *fppf site on* X, X_{fppf} has Sch/X, the category of all schemes over the base scheme X, as its underlying category and the coverings are surjective families of fppf X-morphisms $\{U_i \to U\}$. We denote the fppf site on X by X_{fppf}.

Definition 6.6. A morphism $f : Y \to X$ of schemes is called *quasi-compact* if there exists an Zariski covering $\{V_i \subseteq V\}_{i \in I}$ such that for all $i \in I$ the preimage $f^{-1}(V_i)$ is quasi-compact (as a topological space).

Definition 6.7. We say a map of schemes $f : Y \to X$ is *fpcq* (stands for fidèlement plat et quasi-compact) if it is faithfully flat and for every $y \in Y$ there is an open neighborhood $U \ni y$ such that $f(U)$ is open and the restriction $f|_U : U \to f(U)$ is quasi-compact.

Equivalently we can say that $f : Y \to X$ is fpqc if it is faithfully flat and if there exists a covering

$$X = \bigcup U_i$$

of open affine subschemes, such that each U_i is the image of an open quasi-compact subset of Y.

Example 6.8 (The fpqc site)**.** Let X be a scheme. The (large) *fpqc site on X* has as its underlying category $\mathbf{Sch/X}$ and coverings are given by families of morphisms $\{f_i : U_i \to U\}_{i \in I}$ each of which is flat and such that there exists a natural number $n \geq 0$, function

$$a : \{1, \ldots, n\} \to I$$

and affine open subsets $V_j \in U_{a(j)}$ for all $j \in \{1, \ldots, n\}$ such that

$$\bigcup_{j=1}^{n} f_{a(j)}(V_j) = U$$

We denote the fpqc site on X by X_{fpqc}.

Remark 6.9. Any detailed study of the fpqc site is beyond the scope of this thesis. In particular we shall not study set-theoretical difficulties relating to sheafification of fpqc presheaves.

We consider here the relationship between X_{fppf} and X_{fpqc}.

Proposition 6.10. *Let $U \to X$ be a scheme over X . Any fppf covering of U is an fpqc covering.*

Proof. See for example Stacks Project [2017, Tag 0DFQ] $\square$

Example 6.11 (Site associated with a profinite group)**.** Let π be a profinite group. The *site associated with* π is denoted by $\mathbb{T}_\pi$. It has as its underlying category $\pi-\mathrm{Rep}_\mathrm{c}$ and any family $\{U_i \to U\}$ of surjective $\pi-$maps is a covering.

Definition 6.12. Similarly to the definition of the category $\mathbf{F\acute{E}t}_X$, we define the category $\mathbf{\acute{E}t}_X$ by dropping the finiteness condition. That is, for a given scheme X the category $\mathbf{\acute{E}t}_X$ is the full subcategory of $\mathbf{Sch/X}$ consisting of étale schemes Y over X

Example 6.13 (The étale site)**.** Let X be a scheme. The (small) *étale site on X* has as its underlying category $\mathbf{\acute{E}t}_X$ and coverings are taken to be surjective families $\{U_i \to U\}$ of étale morphisms of $X-$schemes. We denote it by $X_{\text{ét}}$.

Now we consider the relationship between the Zariski site on X and the étale site on X for some given scheme X.

It is clear that since Zariski coverings $\{f_i : U_i \to U\}_{i \in I}$ consist of inclusions, that are in particular open immersions and we know that open immersions are étale, the following proposition holds.

Proposition 6.14. *Let $U \to X$ be a scheme over X. Any Zariski covering of U is an étale covering.*

We can furthermore look at the relationship between the étale site on X and the fppf site on X, for some given scheme X. The following proposition is clear.

Proposition 6.15. *Let $U \to X$ be an étale scheme over X. Any étale covering $\{f_i : U_i \to U\}$ is an fppf covering.*

To summarize, Zariski coverings are étale, étale coverings are fppf and fppf coverings are fpqc.

This is important for (at least) two reasons, firstly as we have theorems that tell us that sometimes we can check the sheaf property of a presheaf on a smaller class of coverings. And secondly because this allows us to use results from the theory of faithfully flat descent.

Remark 6.16. What we call here the étale site on X is more correctly called the *small étale site on* X. There is a big étale site that has as it's underlying category the whole of Sch/X. However we shall not use this site here and thus unambiguously use the term étale site for the small étale site. Similarly we have introduced as examples the large fppf and fpqc sites and there are corresponding *small* sites that we do not look at here.

Let X be a topological space and $\mathcal{D}$ some category. Then a $\mathcal{D}-$valued presheaf on X is simply a contravariant functor $\mathcal{F} : op(X) \to \mathcal{D}$. This definition carries over naively to presheaves on any category $\mathcal{C}$. However the glueing conditions that define a sheaf, depend vitally on the topology. The notion of a Grothendieck topology allows us to define sheaves on sites, i.e. it allows us to formulate the appropriate "glueing conditions".

Definition 6.17. Let $\mathcal{S} = (\mathcal{C}, \mathcal{T})$ be a site and $\mathcal{D}$ a category. A $\mathcal{D}-$*valued presheaf on* $\mathcal{S}$ is a contravariant functor

$$\mathcal{F} : \mathcal{C} \to \mathcal{D}$$

A presheaf is called a *sheaf* if it satisfies the condition that

$$\mathcal{F}(U) \longrightarrow \prod_{i \in I} \mathcal{F}(U_i) \rightrightarrows \prod_{(i,j) \in I \times I} \mathcal{F}(U_i \times_U U_j)$$

is exact for every covering $\{U_i \to U\}$ in $\mathcal{T}$.

A *morphism of presheaves* is just a natural transformation of the functors, and a *morphism of sheaves* is a morphism of presheaves of the sheaves.

If we look at this sheaf condition of the site associated with a topological space X, then the fibre product is simply the intersection and the condition translates directly to the classical uniqueness and gluing conditions.

6.2. **Étale sheaves.** Now we shall look more closely at sheaves on the site $\acute{\mathbf{E}}\mathbf{t}_X$.

Definition 6.18. Let R be a ring and X a scheme. Then an étale presheaf of R-modules on X is a contravariant functor

$$\mathcal{F} : \acute{\mathbf{E}}\mathbf{t}_X \to R - \mathrm{Mod}$$

When $R = \mathbb{Z}$ we will simply say that $\mathcal{F}$ is an étale presheaf on X.

As we saw in the previous section, an étale presheaf of R-modules is called an *étale sheaf* of R-modules if for any étale map $U \to X$ and any étale covering $\{u_i : U_i \to U\}_{i\in I}$ the following sequence is exact

$$\mathcal{F}(U) \longrightarrow \prod_{i\in I} \mathcal{F}(U_i) \rightrightarrows \prod_{(i,j)\in I\times I} \mathcal{F}(U_i \cap U_j)$$

The following example of an étale sheaf is so important that we present it as a definition rather than an example.

Definition 6.19. Let M be an R-module and X a scheme. For any étale map $U \to X$ we denote by $\pi_0(U)$ the (finite by our assumptions) set of connected components of U. The assignment

$$U \mapsto M^{\pi_0(U)}$$

defines a presheaf of R-modules on $X_{\text{ét}}$, called the constant presheaf on $X_{\text{ét}}$ associated with M.

Definition 6.20. Let $\bar{x} : \operatorname{Spec}\Omega \to X$ be a geometric point. An étale neighborhood U of $\overline{x}$ is an étale $U \to X$ such that there is a map $\operatorname{Spec}\Omega \to U$ making the following diagram commutes

$$\begin{tikzcd}
 & U \arrow[d] \\
\operatorname{Spec}\Omega \arrow[ur] \arrow[r, "\overline{x}"] & X
\end{tikzcd}$$

if no confusion can arise we simply say that U is an étale neighborhood of $\overline{x}$ and omit mentioning the maps $\operatorname{Spec}\Omega \to U$ and $U \to X$.

There is a natural notion of morphisms of étale neighborhoods, namely if U_1 and U_2 are two étale neighborhood of the geometric point $\overline{x}$ then a morphism of schemes $\phi : U_1 \to U_2$ is called a morphism of étale neighborhoods of $\overline{x}$ if the following diagram commutes

$$\begin{tikzcd}
U_1 \arrow[rr, "\phi"] \arrow[ddr] & & U_2 \arrow[ddl] \\
 & \operatorname{Spec}\Omega \arrow[ul] \arrow[ur] \arrow[d, "\overline{x}"] & \\
 & X &
\end{tikzcd}$$

We have the notion of stalks like in classical sheaf theory.

Definition 6.21. Let $\overline{x} : \operatorname{Spec}\Omega \to X$ be a geometric point and $\mathcal{F}$ a presheaf of R-modules on $X_{\text{ét}}$. Then we define the *stalk of* $\mathcal{F}$ *at* $\overline{x}$ as

$$\mathcal{F}_{\overline{x}} := \varinjlim_{\overline{x}\to U} \mathcal{F}(U)$$

where the limit is taken over all (isomorphism classes of) étale neighborhoods of $\overline{x}$.

Definition 6.22. Let X be a scheme and $\mathcal{F}$ an étale sheaf on X. Then $\mathcal{F}$ is called a *skyscraper sheaf* if the stalk $\mathcal{F}_{\overline{x}}$ is zero except for a finite number of points $x \in X$.

Let us now look at some examples of étale sheaves on a scheme X.

Example 6.23. We begin by looking at a the construction of sheaf, associated with an R-module. Let X be a scheme and M an R-module and $\overline{x}$ be a geometric point in X. For any étale map $\phi : U \to X$ we define

$$M^{\overline{x}}(U) = \bigoplus_{\operatorname{Hom}_X(\overline{x},U)} M$$

This defines an étale sheaf of R-modules on X and we have a natural isomorphism

$$\operatorname{Hom}(\mathcal{F}, M^{\overline{x}}) \xrightarrow{\cong} \operatorname{Hom}_R(\mathcal{F}_{\overline{x}}, M)$$

Remark 6.24. Notice that the sheaf $M^{\overline{x}}$ defined above, needs not be a skyscraper sheaf. It is a skyscraper sheaf when the image x of the geometric point $\overline{x}$ is a closed point in X, but it can fail to be when x is not a closed point.

Definition 6.25. Given a scheme with the étale topology $X_{\text{ét}}$ the natural notions of morphisms between (pre-)sheaves make the collection of (pre-)sheaves a category. We denote the category of étale presheaves on X by

$$\operatorname{Psh}(X_{\text{ét}})$$

and the category of étale sheaves on X by

$$\operatorname{Sh}(X_{\text{ét}})$$

Notice that there is a natural notion of addition of morphisms of (pre-)sheaves. Namely, let $\psi, \phi \in \operatorname{Hom}_{\operatorname{Psh}_R(X_{\text{ét}})}(\mathcal{P}_1, \mathcal{P}_2)$ then we can define the morphism $\psi + \phi \in \operatorname{Hom}_{\operatorname{Psh}_R(X_{\text{ét}})}(\mathcal{P}_1, \mathcal{P}_2)$ by

$$(\psi + \phi)(U) := \psi(U) + \phi(U)$$

i.e. we add on the level of R-modules. This endowes the Hom-sets with the structure of Abelian groups such that composition of maps distributes over addition.

We now investigate these categories further and show that they are Abelian categories.

As in the classical theory of sheaves, there is a *sheafification functor*

$$\widehat{\ }: \mathrm{Psh}(X_{\text{ét}}) \to \mathrm{Sh}(X_{\text{ét}})$$

that associates a sheaf with any presheaf. It is the adjoint of the forgetful functor

$$For : \mathrm{Sh}(X_{\text{ét}}) \to \mathrm{Psh}(X_{\text{ét}})$$

that forgets the local properties of a sheaf.

That means that for any presheaf $\mathcal{P}$ the associated sheaf of $\mathcal{P}$ is a sheaf $\widehat{\mathcal{P}}$ and a morphism of presheaves

$$i : \mathcal{P} \to \widehat{\mathcal{P}}$$

such that for any étale sheaf $\mathcal{F}$ on X any morphisms of presheaves $\alpha : \mathcal{P} \to \mathcal{F}$ factors uniquely through i, i.e there is a unique homomorphisms of presheaves $\alpha^{'} : \widehat{\mathcal{P}} \to \mathcal{F}$ making the following diagram commutative

$$\begin{tikzcd} \mathcal{P} \arrow[r, "i"] \arrow[dr, "\alpha"'] & \widehat{\mathcal{P}} \arrow[d, dashed, "\exists ! \alpha^{'}"] \\ & \mathcal{F} \end{tikzcd}$$

When we need to be explicit, we say that $(\widehat{\mathcal{P}}, i)$ is the sheaf associated with $\mathcal{P}$.

Proposition 6.26. *For every presheaf $\mathcal{P}$ on $X_{\text{ét}}$, there exists an associated sheaf $i : \mathcal{P} \to \widehat{\mathcal{P}}$ such that the map i induced isomorphisms on stalks*

$$i_{\overline{x}} : \mathcal{P}_{\overline{x}} \to (\widehat{\mathcal{P}})_{\overline{x}}$$

The functor

$$\widehat{\ }: \mathrm{Psh}(X_{\text{ét}}) \to \mathrm{Sh}(X_{\text{ét}})$$

is exact.

Proof. See Milne [2013, Thm. 7.15] □

We now introduce the neccesary constructions and operations needed on the category $\mathrm{Psh}_R(X_{\text{ét}})$ and see that it is an Abelian category. Although we are considering here specifically the case of $R - \mathrm{Mod}$ it is proved in essentially the same way that the category of $\mathcal{D}$-valued presheaves on a category $\mathcal{C}$ (that has products) is Abelian if $\mathcal{D}$ itself is Abelian (since all the relevant objects and operations are defined on the $\mathcal{D}$-level).

Consider as before, a scheme X and a ring R. There is a natural notion of the *zero presheaf* in $\mathrm{Psh}_R(X_{\text{ét}})$ that simply sends any étale map $\phi : U \to X$ to the trivial R-module. This is the zero object in the category $\mathrm{Psh}_R(X_{\text{ét}})$.

Now we define the kernel and the cokernel of a map of presheaves of R-modules on $X_{\text{ét}}$.

Definition 6.27. Let X be a scheme and $\mathcal{P}_1, \mathcal{P}_2 \in \mathrm{Psh}_R(X_{\text{ét}})$. If

$$\psi : \mathcal{P}_1 \to \mathcal{P}_2$$

is a map of presheaves, then we define the *kernel* of ψ to be the presheaf given by

$$\ker(\psi)(U) := \ker(\psi(U) : \mathcal{P}_1(U) \to \mathcal{P}_2(U))$$

along with the canonical monomorphism $\ker(\psi) \hookrightarrow \mathcal{P}$. Similarly we define the *cokernel* of ψ to be the presheaf given by

$$\mathrm{coker}(\psi)(U) := \mathrm{coker}(\psi(U) : \mathcal{P}_1(U) \to \mathcal{P}_2(U))$$

along with the canonical epimorphism $\mathcal{P}_2 \twoheadrightarrow \mathrm{coker}(\psi)$.

Definition 6.28. Let $\mathcal{P}_1$ and $\mathcal{P}_2$ be étale presheaves of R-modules on a scheme X. We define the direct sum of $\mathcal{P}_1$ and $\mathcal{P}_2$ to be the presheaf $\mathcal{P}_1 \oplus \mathcal{P}_2 \in \mathrm{Psh}_R(X_{\text{ét}})$ defined by

$$(\mathcal{P}_1 \oplus \mathcal{P}_2)(U) := \mathcal{P}_1(U) \oplus \mathcal{P}_2(U)$$

for all $U \in \mathbf{\acute{E}t}_X$.

It's clear that in this manner we obtain a presheaf. This direct sum is the coproduct in $\mathrm{Psh}_R(X_{\text{ét}})$.

With the kernels, cokernels, zero object and coproduct as defined above we get a structure of an Abelian category on both the category of presheaves of R-modules on $X_{\text{ét}}$ and the category of sheaves of R-modules on $X_{\text{ét}}$.

Proposition 6.29. *Let X be a scheme and R a ring. The category $\mathrm{Psh}_R(X_{\text{ét}})$ of étale presheaves of R-modules on X, is an Abelian category. Furthermore the full subcategory $\mathrm{Sh}_R(X_{\text{ét}})$ of étale sheaves of R-modules on X is an Abelian category.*

Proof. See Milne [2013, Chap. I §7, more specifically prop. 7.8] $\square$

The following proposition tells us that we can check whether an étale presheaf on a scheme X is a sheaf by considering Zariski coverings.

Proposition 6.30. *Let X be a scheme and let $\mathcal{F}$ be a presheaf on $X_{\text{ét}}$. Then $\mathcal{F}$ is a sheaf on $X_{\text{ét}}$ if and only if it satisfies the sheaf property in the following two cases:*

(1) $\{U_i \to U\}_{i \in I}$ is a Zariski open covering.
(2) $\{V \to U\}$ is an étale covering consisting of a single map, where both V and U are affine.

Proof. See for example Milne [2013, Prop. 6.6] □

So far we have only looked at operations on (pre-)sheaves on a fixed étale site. We want to be able to move from one étale site to another, and the two most important ways of doing that is via the direct image and inverse image functors associated with a morphisms of schemes.

Definition 6.31. Let $f : Y \to X$ be a morphisms of schemes, and let R be a ring. Then we define the direct image functor of presheaves of R-modules associated with f

$$f_* : \mathrm{Psh}_R(Y_{\text{ét}}) \to \mathrm{Psh}_R(X_{\text{ét}})$$
$$\mathcal{F} \mapsto f_*\mathcal{F}$$

by

$$f_*\mathcal{F}(V) = \mathcal{F}(V \times_X Y)$$

for any étale $V \to X$.

Remark 6.32. Notice that the stability under base chance of the étale property assures us that this definition makes sense.

Before we consider what happens when we restrict ourselves to direct images of étale *sheaves* on Y, we proof the exactness of the direct image functor on presheaves.

Proposition 6.33. *Let $f : Y \to X$ be a morphisms of schemes and R be a ring. Then the direct image functor*

$$f_* : \mathrm{Psh}_R(Y_{\text{ét}}) \to \mathrm{Psh}_R(X_{\text{ét}})$$

is exact.

Proof. Consider the exact sequence of étale presheaves of R-modules on Y

$$0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$$

and look at the image of it

$$0 \to f_*\mathcal{F}_1 \to f_*\mathcal{F}_2 \to f_*\mathcal{F}_3 \to 0$$

under the direct image functor f_*. To prove this is exact, we prove that the sequence of R-modules

$$0 \to f_*\mathcal{F}_1(V) \to f_*\mathcal{F}_2(V) \to f_*\mathcal{F}_3(V) \to 0$$

is exact for any étale $V \to X$. But by definition this is

$$0 \to \mathcal{F}_1(V \times_X Y) \to \mathcal{F}_2(V \times_X Y) \to \mathcal{F}_3(V \times_X Y) \to 0$$

which is exact by assumption. $\square$

Proposition 6.34. *Let $f : Y \to X$ be a morphism of schemes and let R be a ring. If $\mathcal{F} \in \mathrm{Sh}_R(Y_{\text{ét}}) \subseteq \mathrm{Psh}_R(Y_{\text{ét}})$ then $f_*\mathcal{F} \in \mathrm{Sh}_R(X_{\text{ét}})$.*

Proof. See Milne [2013, Lem. 8.1] $\square$

This shows that f_* restricts to a functor

$$f_* : \mathrm{Sh}_R(Y_{\text{ét}}) \to \mathrm{Sh}_R(X_{\text{ét}})$$

We shall mean this restricted functor when we talk of the direct image from here on.

Notice that left-exactness of sheaves can be checked on sections, and so the proof of exactness of f_* for presheaves gives us that f_* is left exact as a functor of étale sheaves of R-modules. It is however not right-exact in general.

Now consider a geometric point $\overline{y} : \operatorname{Spec} \Omega \to Y$. By composing with a map of schemes $f : Y \to X$ we obtain a geometric point in X, denoted by $f(\overline{y})$. If $\mathcal{F}$ is an étale sheaf on Y the stalk of $f_*\mathcal{F}$ at $f(\overline{y})$ is given by

$$(f_*\mathcal{F})_{f(\overline{y})} = \varinjlim f_*\mathcal{F}(V)$$

where the limit is over all étale neighborhoods V of $f(\overline{y})$. Equivalently we can write the stalk as

$$(f_*\mathcal{F})_{f(\overline{y})} = \varinjlim \mathcal{F}(V \times_X Y)$$

and the limit is over all étale neighborhoods of $\overline{y}$ of the form $V \times_X Y$ for some étale $V \to X$.

This defines a canonical map

$$(f_*\mathcal{F})_{f(\overline{y})} \to \mathcal{F}_{\overline{y}}$$

which is in general neither surjective nor injective.

Definition 6.35. Let $f : Y \to X$ be a morphism of schemes and let R be a ring. The left exactness of the direct image functor

$$f_* : \mathrm{Sh}_R(Y_{\text{ét}}) \to \mathrm{Sh}_R(X_{\text{ét}})$$

implies that it admits a left adjoint. This functor

$$f^* : \mathrm{Sh}_R(X_{\text{ét}}) \to \mathrm{Sh}_R(Y_{\text{ét}})$$

is called the inverse image functor associated with f. By the definition of adjoint functors we have for any $\mathcal{G} \in \mathrm{Sh}_R(Y_{\text{ét}})$ and $\mathcal{F} \in \mathrm{Sh}_R(X_{\text{ét}})$

$$\mathrm{Hom}_{Y_{\text{ét}}}(f^*\mathcal{F}, \mathcal{G}) = \mathrm{Hom}_{X_{\text{ét}}}(\mathcal{F}, f_*\mathcal{G})$$

The stalk of the inverse image functor at a geometric point $\overline{x}$ is as expected the same as the stalk of the original sheaf at the image of that point. That is, if $f : Y \to X$ is a morphism of schemes, R a ring and $\mathcal{G} \in \mathrm{Sh}_R(X_{\text{ét}})$ and if $\overline{y}$ is a geometric point in Y and we denote by $f(\overline{y})$ the geometric point of X given by the composition $\overline{y} \to Y \to X$ then

$$(f^*\mathcal{G})_{\overline{y}} = \mathcal{G}_{f(\overline{y})}$$

Another classical operation on sheaves we wish to define in the étale context is *extension by zero*.

Definition 6.36. Let $j : U \to X$ be an open immersion and let $i : Z \to X$ be the closed immersion of the comlement of U. If $\mathcal{F}$ is a sheaf of R-modules on U then the extension by zero of $\mathcal{F}$ is defined as

$$j_!\mathcal{F} := \ker(j_*\mathcal{F} \to i_*i^*j_*\mathcal{F})$$

where the map $j_*\mathcal{F} \to i_*i^*j_*\mathcal{F}$ is the natural adjunction map for the adjoint pair (i_*, i^*).

The stalk of the extension by zero of a sheaf at a geometric point $\overline{x}$, is as expected the same as the stalk of the original sheaf if $\overline{x}$ lies over a point $x \in U$ and zero otherwise,

$$(j_!\mathcal{F})_{\overline{x}} = \begin{cases} \mathcal{F}_{\overline{x}} & \text{if } x \in U \\ 0 & \text{otherwise} \end{cases}$$

Furthermore $j_!$ is an exact functor from the category of étale sheaves of R-modules on U to the category of étale sheaves of R-modules on X that is left adjoint to j^* so we have a functorial isomorphism

$$\mathrm{Hom}(j_!\mathcal{G}, \mathcal{F}) \cong \mathrm{Hom}(\mathcal{G}, j^*\mathcal{F})$$

for $\mathcal{G} \in \mathrm{Sh}_R(U_{\text{ét}})$ and $\mathcal{F} \in \mathrm{Sh}_R(X_{\text{ét}})$.

Definition 6.37. Let X be a scheme, R a ring and M an R-module. We previously showed how to obtain a presheaf of R-modules on $X_{\text{ét}}$ associated with M by

$$U \mapsto M^{\pi_0(U)}$$

for any étale $U \to X$. The associated sheaf of this presheaf is called the constant sheaf on X associated with M and is denoted by $\underline{M}_X$.

We say that an étale sheaf $\mathcal{F}$ of R-modules on X is *locally constant* if there exists an étale covering $\{U_i \to X\}_{i \in I}$ of X such that

$$\mathcal{F}|_{U_i} \cong \underline{M_i}_{U_i}$$

Where each M_i is an R-module.

Finally, we say that a sheaf $\mathcal{F}$ is a *local system* of R-modules if it is locally constant, and has finite fibres over each point. We denote by $\mathrm{Loc}_R(X_{\text{ét}})$ the full subcategory of $\mathrm{Sh}_R(X_{\text{ét}})$ whose objects are local systems.

6.3. ***l*-adic sheaves.** In this subsection we want to prove an analogue of the theorem in topology that gives us an equivalence between the category of local systems of $\mathbb{C}$-vector spaces on a topological space X (assumed to be connected, locally path-connected and semi-locally simply connected), and the category of finite dimensional complex representations of the fundamental group $\pi_1(X)$.

Ideally we would want to prove this with $\mathbb{C}$ replaced by the field $\mathbb{Q}_l$ or $\bar{\mathbb{Q}}_l$. However, this fails in general and we will give a concrete example of that.

To remidy this failure, one can look in one of two directions.

(1) Change our concept of a local system, or
(2) change our definition of the fundamental group (or more correctly, change the site we are interested in and thereby changing our definition of the fundamental group, but also our definition of cohomology etc.).

Grothendieck and his school went the route of changing the concept of a local system, by introducing lisse sheaves, which are certain projective systems of sheaves. We consider this approach in this chapter.

Bhatt and Scholze have gone the other route, by modifying the site and considering the *pro-étale site and the pro-étele topos* where our intuitive definition of a local system works. This is the subject of chapter 7.

To motivate the construction of l-adic sheaves we look at an example where we do not have an equivalence of categories between local systems of $\bar{\mathbb{Q}}_l$-modules and $\bar{\mathbb{Q}}_l$ representations of the fundamental group.

Example 6.38. Recall from 5.62 the nodal curve X obtained from $\mathbb{P}^1$ by gluing 0 and ∞ together transversally over some algebraically closed field k. We let $\overline{x} : \operatorname{Spec} k \to X$ be a geometric point with the node as image.

We saw that the finite étale covers are Y_n which is a chain of n copies of $\mathbb{P}^1$ such that the ∞ point of the i-th copy of $\mathbb{P}^1$ is glued to the 0 of the $i+1$-st copy, and this is done cyclically. Now we could do the same procedure to infinity instead of cyclically, that is; we could look at an infinite chain of copies of $\mathbb{P}^1$ glued as before. Let us call this Y_∞. This is obviously not a finite étale cover because $g : Y_\infty \to X$ is not a finite morphism, it is however still étale.

The trivial $\mathbb{Q}_l$-local system $\underline{\mathbb{Q}_l}_{Y_\infty}$ descends to a rank 1 $\mathbb{Q}_l$-local system $\mathcal{L}$ on X where the descent data ϕ is given by identifying the fibres over 0 and ∞ by multiplication by l.

Now if we assume that we have

$$\mathrm{Loc}_{\mathbb{Z}_l}(X_{\text{ét}}) \cong \pi_1^{\text{ét}}(X, \overline{x}) - \mathrm{Rep_c}^{\mathbb{Z}_l}$$

and

$$\mathrm{Loc}_{\mathbb{Q}_l}(X_{\text{ét}}) \cong \pi_1^{\text{ét}}(X, \overline{x}) - \mathrm{Rep_c}^{\mathbb{Q}_l}$$

then this local system $\mathcal{L}$ should correspond to a representation

$$\alpha : \pi_1^{\text{ét}}(X, \overline{x}) \to GL_1(\mathbb{Q}_l) = \mathbb{Q}_l \setminus \{0\}$$

Now this is a continuous map and $\pi_1^{\text{ét}}(X, \overline{x})$ is compact (profinite groups are compact) i.e. the image should be a compact subgroup of $\mathbb{Q}_l \setminus \{0\}$. This implies that it must lie entirely inside punctured closed unit disc, i.e. inside $\mathbb{Z}_l \setminus \{0\}$. Every element in the image is a unit, so finally we have

$$\mathrm{im}(\alpha) \subseteq GL_1(\mathbb{Z}_l) = \mathbb{Z}_l^\times$$

By assumption this corresponds to a rank 1 $\mathbb{Z}_l$-local system $\mathcal{L}'$ on X and so

$$\mathcal{L} = \mathcal{L}' \otimes_{\mathbb{Z}_l} \mathbb{Q}_l$$

Again by descent this corresponds to a $\mathbb{Z}_l$-sheaf $\mathcal{F}$ with descent data ψ on Y_∞ such that

$$(\underline{\mathbb{Q}_l}_{Y_\infty}, \phi) \cong (\mathcal{F} \otimes_{\mathbb{Z}_l} \mathbb{Q}_l, \psi)$$

in the category of sheaves of $\mathbb{Q}_l$-vector spaces with descent data on Y_∞. But this cannot hold since multiplication by l does not give an isomorphism of $\mathbb{Z}_l$-modules, since l is not a unit in $\mathbb{Z}_l$.

Definition 6.39. Let X be a scheme. A subset $Z \subseteq X$ is called *locally closed* if it is an intersection of an open subset of X with a closed subset of X. We endow such a Z with the reduced scheme structure and obtain an immersion

$$i : Z \hookrightarrow X$$

Definition 6.40. Let X be a scheme and let R be a torsion ring, e.g. $\mathbb{Z}/n\mathbb{Z}$ for some $n \in \mathbb{N}$. A sheaf $\mathcal{F}$ of R-modules on $X_{\text{ét}}$ is called *constructible* if there exist locally closed subsets $X_1, \ldots, X_r$ of X such that

- $X = \coprod_i X_i$ and,
- $\mathcal{F}|_{X_i}$ is locally constant with stalks that are finitely generated R-modules.

Definition 6.41. Let X be a scheme, and denote by $\mathfrak{m}$ the unique maximal ideal in $\mathbb{Z}_l$.

(1) A *constructible* $\mathbb{Z}_l$-sheaf, or an l-adic sheaf on X is a projective system

$$\mathcal{F} := (\mathcal{F}_n)_{n\geq 1}$$

of sheaves of $\mathbb{Z}_l$-modules on $X_{\text{ét}}$ such that
  (i) $\mathfrak{m}^n\mathcal{F}_n = 0$ and $\mathcal{F}_n$ is a constructible sheaf of $\mathbb{Z}_l/\mathfrak{m}^n$-modules on $X_{\text{ét}}$
  (ii) $\mathcal{F}_n = \mathcal{F}_{n+1} \otimes_{\mathbb{Z}_l/\mathfrak{m}^{n+1}} \mathbb{Z}_l/\mathfrak{m}^n$, for all $n \geq 1$.

(2) A *lisse* $\mathbb{Z}_l$-sheaf on X, is an l-adic sheaf $\mathcal{F} = (\mathcal{F}_n)$ such that each $\mathcal{F}_n$ is a locally constant sheaf of $\mathbb{Z}_l/\mathfrak{m}^n$-modules.

As usual, when we define some new objects, we want to define morphisms betweeen them.

Definition 6.42. Let X be a scheme and consider two l-adic sheaves $\mathcal{F} = (\mathcal{F}_n)$ and $\mathcal{G} = (\mathcal{G}_n)$ on X. From condition (ii) in the definition of l-adic sheaves we clearly have a homomorphism of sheaves of $\mathbb{Z}_l$-modules

$$\begin{aligned} \operatorname{Hom}(\mathcal{F}_{n+1}, \mathcal{G}_{n+1}) &\to \operatorname{Hom}(\mathcal{F}_n, \mathcal{G}_n) \\ \phi &\mapsto \phi \otimes \mathbb{Z}_l/\mathfrak{m}^n \end{aligned}$$

and we define the category of l-adic sheaves on X by declaring the morphisms to be given by

$$\operatorname{Hom}(\mathcal{F}, \mathcal{G}) = \varprojlim_n \operatorname{Hom}(\mathcal{F}_n, \mathcal{G}_n)$$

We denote the category of l-adic sheaves on X by $\mathrm{Sh}^{con}_{\mathbb{Z}_l}(X)$.

We define the category of lisse sheaves on X to be the full subcategory of $\mathrm{Sh}^{con}_{\mathbb{Z}_l}(X)$ whose objects are lisse.

We shall denote the category of lisse sheaves on X by $\mathrm{Sh}^{lis}_{\mathbb{Z}_l}(X)$

We have the following proposition.

Proposition 6.43. *Given a scheme X, the categories $\mathrm{Sh}^{con}_{\mathbb{Z}_l}(X)$ and $\mathrm{Sh}^{lis}_{\mathbb{Z}_l}(X)$ are both Abelian categories.*

Proof. $\mathrm{Sh}^{lis}_{\mathbb{Z}_l}(X)$ is a full subcategory of $\mathrm{Sh}^{con}_{\mathbb{Z}_l}(X)$ that is clearly closed under taking kernels and cokernels, so it suffices to prove that $\mathrm{Sh}^{con}_{\mathbb{Z}_l}(X)$ is Abelian. This can be found in Grothendieck et al. [1977, Exposé VI, prop. 1.1.3] $\square$

Definition 6.44. Let X be a scheme, $\overline{x} : \mathrm{Spec}\,\Omega \to X$ be a geometric point in X and let l be a prime number that is invertible on X.Let $\mathcal{F} = (\mathcal{F}_n)_{n\geq 1}$ be a lisse $\mathbb{Z}_l$-sheaf on X. The stalk of $\mathcal{F}$ at $\overline{x}$ is defined as

$$\mathcal{F}_{\overline{x}} := \varprojlim_n \mathcal{F}_{n,\overline{x}}$$

Lemma 6.45. *Consider the finite ring $\mathbb{Z}/l^n\mathbb{Z}$ where l is a prime and $n \geq 1$. Let X be a scheme on which l is invertible, $\overline{x} : \mathrm{Spec}\,\Omega \to X$ be a geometric point in X and let $\mathrm{Loc}_{\mathbb{Z}/l^n\mathbb{Z}}(X_{\text{ét}})$ denote the category of locally constant constructible sheaves of $\mathbb{Z}/l^n\mathbb{Z}$-modules on $X_{\text{ét}}$. Then we have an equivalence of categories*

$$\begin{aligned} \mathrm{Loc}_{\mathbb{Z}/l^n\mathbb{Z}}(X_{\text{ét}}) &\xrightarrow{\cong} \pi_1^{\text{ét}}(X,\overline{x}) - \mathrm{Rep_c}^{\mathbb{Z}/l^n\mathbb{Z}} \\ \mathcal{F} &\mapsto \mathcal{F}_{\overline{x}} \end{aligned}$$

where $\pi_1(X,\overline{x}) - \mathrm{Rep_c}^{\mathbb{Z}/l^n\mathbb{Z}}$ denotes the category of finitely generated $\mathbb{Z}/l^n\mathbb{Z}$-modules M with the structure of a continuous representation of $\pi_1^{\text{ét}}(X,\overline{x})$.

Proof. From the construction of the étale fundamental group we know that there exists an inverse system (P_α) of finite étale covers for which we have geometric points $\mathrm{Spec}\,\Omega \to P_\alpha$ such that the diagram

$$\begin{tikzcd} & P_\alpha \arrow[d] \\ \mathrm{Spec}\,\Omega \arrow[ur] \arrow[r, "\overline{x}"'] & X \end{tikzcd}$$

commutes for all α and such that

$$\pi_1(X,\overline{x}) \cong (\varprojlim_\alpha \mathrm{Aut}(P_\alpha)^{op})$$

Now let $\mathcal{F}$ be a locally constant constructible sheaf of $\mathbb{Z}/l^n\mathbb{Z}$-modules on $X_{\text{ét}}$, and let $U \to X$ be an étale neighborhood of $\overline{x}$. We write U_α

for the fibre product

$$U_\alpha := U \times_X P_\alpha \to X$$

for each α, and obtain in this manner an inverse system (U_α) of étale neighborhoods of $\overline{x}$. Any element $\sigma \in \pi_1^{\text{ét}}(X, \overline{x})$ is by construction a compatible system of automorphisms of (P_α) and thus induces a compatible system of automorphisms of (U_α). Therefore $\sigma \in \pi_1^{\text{ét}}(X, \overline{x})$ induces a map

$$\mathcal{F}(U) \to \mathcal{F}(U_\alpha) \xrightarrow{\sigma} \mathcal{F}(U_\alpha) \to \mathcal{F}_{\overline{x}}$$

In the limit over all étale neighborhoods U of $\overline{x}$, σ thus induces a morphism

$$\mathcal{F}_{\overline{x}} \xrightarrow{\sigma^*} \mathcal{F}_{\overline{x}}$$

We have thus from a locally constant constructible sheaf of $\mathbb{Z}/l^n\mathbb{Z}$-modules on $X_{\text{ét}}$ constructed a finitely generated $\mathbb{Z}/l^n\mathbb{Z}$-module with a continuous representation of $\pi_1^{\text{ét}}(X, \overline{x})$. This construction is functorial since if $\psi : \mathcal{F} \to \mathcal{G}$ is a morphism in $\mathrm{Loc}_{\mathbb{Z}/l^n\mathbb{Z}}(X_{\text{ét}})$ then we obtain a commutative diagram by definition

$$\begin{tikzcd}
\mathcal{F}(U) \arrow[r] \arrow[d, "\psi(U)"'] & \mathcal{F}(U_\alpha) \arrow[r, "\sigma^*"] \arrow[d, "\psi(U_\alpha)"'] & \mathcal{F}(U_\alpha) \arrow[r] \arrow[d, "\psi(U_\alpha)"'] & \mathcal{F}_{\overline{x}} \arrow[d, "\psi_{\overline{x}}"'] \\
\mathcal{G}(U) \arrow[r] & \mathcal{G}(U_\alpha) \arrow[r, "\sigma^*"'] & \mathcal{G}(U_\alpha) \arrow[r] & \mathcal{G}_{\overline{x}}
\end{tikzcd}$$

So now we have obtained a functor

$$\mathrm{Loc}_{\mathbb{Z}/l^n\mathbb{Z}}(X_{\text{ét}}) \to \pi_1^{\text{ét}}(X, \overline{x}) - \mathrm{Rep_c}^{\mathbb{Z}/l^n\mathbb{Z}}$$

Let us now construct an inverse functor. Let M be a finitely generated $\mathbb{Z}/l^n\mathbb{Z}$-module carrying a continuous representation of $\pi_1^{\text{ét}}(X, \overline{x})$. The group $\mathrm{Aut}_{\mathbb{Z}/l^n\mathbb{Z}}(M)$ is finite so there exists a finite étale Galois cover $f : P_\alpha \to X$, with Galois group G, such that the representation $\pi_1^{\text{ét}}(X, \overline{x}) \to \mathrm{Aut}_{\mathbb{Z}/l^n\mathbb{Z}}(M)$ factors over G. The constant sheaf M_{P_α} has a G-action given by the map

$$M_{P_\alpha}(\sigma) : \sigma_* M_{P_\alpha} \to M_{P_\alpha}$$

given by

$$M_{P_\alpha}(V \times_{P_\alpha} P_\alpha) = M \xrightarrow{\sigma^{-1}} M = M_{P_\alpha}(V)$$

where V is connected, $V \to P_\alpha$ is étale, and the fibre product is

$$\begin{tikzcd}
V \times_{P_\alpha} P_\alpha \arrow[r] \arrow[d] & P_\alpha \arrow[d, "\sigma"] \\
V \arrow[r] & P_\alpha
\end{tikzcd}$$

Now we have a sheaf with a G-action M_{P_α} on P_α and the action is compatible with the map $f : P_\alpha \to X$ in the sense that for any $\sigma \in G$ the following diagram commutes

$$\begin{tikzcd} P_\alpha \arrow[rr, "\sigma"] \arrow[dr, "f"'] & & P_\alpha \arrow[dl, "f"] \\ & X & \end{tikzcd}$$

Therefore we get an induced map

$$f : G \setminus P_\alpha \to X$$

If we denote the quotient map by $\pi : P_\alpha \to G \setminus P_\alpha$, then π is étale and we obtain via an étale sheaf of $\mathbb{Z}/l^n\mathbb{Z}$-modules on $G \setminus P_\alpha$ by taking the direct image of M_{P_α} by π

$$(M_{P_\alpha})^G := \pi_* M_{P_\alpha}$$

and finally an étale sheaf of $\mathbb{Z}/l^n\mathbb{Z}$-modules on X by taking the direct image of $(M_{P_\alpha})^G$ by the map f

$$\mathcal{F}_M := f_*(M_{P_\alpha})^G$$

Now we claim that $\mathcal{F}_M$ is in $\mathrm{Loc}_{\mathbb{Z}/l^n\mathbb{Z}}(X_{\text{ét}})$. Let $V \to P_\alpha$ be an étale map such that V is connected. We have an isomorphism

$$V \times_X P_\alpha \cong V \times G$$

Any automorphism $id_V \times \sigma$ on $V \times_X P_\alpha$, where $\sigma \in G$, translates via this isomorphism into the automorphism $id_V \times \sigma$ on $V \times G$. Therefore we can write the sections of $\mathcal{F}_M$ over V as

$$\begin{aligned} \mathcal{F}_M(V) &= \{a \in f_*(M_{P_\alpha})(V) | M_{P_\alpha}(\sigma)(a) = a, \forall \sigma \in G\} \\ &= \{(a_\tau) \in M \times G | \sigma^{-1} a_\tau = a_\tau, \forall \sigma, \tau \in G\} \\ &\cong M \end{aligned}$$

The last isomorphism is given by

$$\begin{aligned} &M \to \{(a_\tau) \in M \times G | \sigma^{-1} a_\tau = a_\tau, \forall \sigma, \tau \in G\} \\ &m \mapsto (\sigma m)_\sigma \end{aligned}$$

So $\mathcal{F}_M|_{P_\alpha} \cong M_{P_\alpha}$ and $\mathcal{F}_M$ is a locally constant constructible sheaf of $\mathbb{Z}/l^n\mathbb{Z}$-modules on X.

We see the functorality of this construction by showing that it is independent of the choice of the finite étale Galois cover $f : P_\alpha \to X$. To prove this we assume we are given another such cover $g : P_\beta \to X$. By the same process as above we obtain a sheaf $\mathcal{F}_M'$. Now we know from the theory of Galois categories that there exists a finite étale Galois cover $h : P_\gamma \to X$ that dominates both P_α and P_γ. Again the same construction yields a sheaf $\mathcal{F}''_M$. We have a natural homomorphism of étale sheaves

$$\mathcal{F}_M \to \mathcal{F}''_M$$

that is induced by the map

$$f_* M_{P_\alpha} \to h_*(M_{P_\alpha}|_{P_\gamma}) = h_* M_{P_\gamma}$$

By the above we obtain that

$$\mathcal{F}_M|_{P_\gamma} = M_{P_\gamma} = \mathcal{F}''_M|_{P_\gamma}$$

and the natural homomorphism is an isomorphism étale locally. But then it is an isomorphism of étale sheaves, i.e.

$$\mathcal{F}_M \cong \mathcal{F}''_M$$

Exactly the same procedure establishes the isomorphism

$$\mathcal{F}'_M \cong \mathcal{F}''_M$$

and thus our construction is independent of the choice of $f : P_\alpha \to X$.

We have now constructed two functors in the opposite direction to each other. What remains to show is that they are actually inverse to each other.

First let an $\mathbb{Z}/l^n\mathbb{Z}$-module M, carrying a continuous representation of $\pi_1^{\text{ét}}(X, \overline{x})$ be given. we associate with it the locally constant constructible sheaf of $\mathbb{Z}/l^n\mathbb{Z}$-modules $\mathcal{F}_M$. It is clear by construction that for all "small enough" étale neighborhoods U of $\overline{x}$ we have $\mathcal{F}_M(U) = M$ and thus in the limit we get

$$\mathcal{F}_{M,\overline{x}} = M$$

For the other direction we let $\mathcal{F}$ be a locally constant constructible sheaf of $\mathbb{Z}/l^n\mathbb{Z}$-modules on X. Let $M := \mathcal{F}_{\overline{x}}$ be the associated $\mathbb{Z}/l^n\mathbb{Z}$-module.

Now by proposition there exists a finite étale Galois cover $f : P \to X$ with Galois group G such that

$$f^*\mathcal{F} \cong M_P$$

By definition we have

$$\mathcal{F}_M = f_*(f^*\mathcal{F})^G$$

And we want to show that this is equal to $\mathcal{F}$. We have a natural map induced by adjuntion

$$\mathcal{F} \to f_*(f^*\mathcal{F})^G$$

and it suffices to show that this is an isomorphism when we restrict to P. But $\mathcal{F}_M|_P \cong M_P$ by definition and this agrees with $\mathcal{F}$ on P. $\square$

Now we can prove an analogous equivalence for lisse $\mathbb{Z}_l$-sheaves.

Theorem 6.46. *We let X be a connected, locally Noetherian scheme, $\overline{x} : \operatorname{Spec} \Omega \to X$ a geometric point and let l be a prime that is invertible on X. There are natural equivalences of categories*

$$\mathrm{Sh}_{\mathbb{Z}_l}^{lis}(X) \xrightarrow{\cong} \pi_1^{\text{ét}}(X, \overline{x}) - \mathrm{Rep_c}^{\mathbb{Z}_l}$$

Proof. As before we denote by $\mathfrak{m}$ the unique maximal ideal in $\mathbb{Z}_l$. Let $\mathcal{F}$ be a lisse $\mathbb{Z}_l$-sheaf. Then by definition it is a projective system

$$\mathcal{F} = (\mathcal{F}_n)_{n \geq 1}$$

where each $\mathcal{F}_n$ is a locally constant constructible $\mathbb{Z}_l/\mathfrak{m}^n$-sheaf.

By the previous lemma each $\mathcal{F}_n$ then gives rise to a $\mathbb{Z}_l$-representation of $\pi_1(X, \overline{x})$ that factors through some finite quotient, namely $\mathcal{F}_{n,\overline{x}}$. These representations commute with the transition functions and thus give rise to a continuous representation of $\pi_1(X, \overline{x})$ on the limit

$$\mathcal{F}_{\overline{x}} = \varprojlim_n \mathcal{F}_{n,\overline{x}}$$

which is a finitely generated $\mathbb{Z}_l$-module, see Grothendieck and Dieudonné [1971, Prop. 7.2.9].

This gives us the functor $\mathcal{F} \to \mathcal{F}_{\overline{x}}$. To show that it is an equivalence of categories we construct an inverse functor.

Let M be a $\mathbb{Z}_l$-Representation of $\pi_1^{\text{ét}}(X, \overline{x})$. For each n we obtain the induced representations

$$M_n := M \otimes_{\mathbb{Z}_l} \mathbb{Z}_l/\mathfrak{m}^n$$

and from 6.45 we obtain a corresponding locally constant constructible sheaf of $\mathbb{Z}_l/\mathfrak{m}^n$-modules $\mathcal{F}_{M_n}$. This gives rise to a projective system $\mathcal{F}_M = (\mathcal{F}_{M_n})$ that defines a lisse $\mathbb{Z}_l$-sheaf and it is clear from construction that

$$\mathcal{F}_{M,\overline{x}} = M$$

so this defines the inverse functor. $\square$

Definition 6.47. Let X be a scheme. An l-adic sheaf $\mathbb{F}$ on X is said to be *torsion* if there exists some $n \in \mathbb{N}$ such that

$$l^n : \mathcal{F} \to \mathcal{F}$$

is the zero map.

We now define the category of lisse $\mathbb{Q}_l$-sheaves by localization. First we recall the definition of localization in a category.

Definition 6.48. Let $\mathcal{C}$ be a category, and $\mathcal{M}$ be some class of morphisms in $\mathcal{C}$. Then the localization of $\mathcal{C}$ with respect to $\mathcal{M}$ is a category $\mathcal{C}[\mathcal{M}^{-1}]$ and a canonical localization functor

$$L : \mathcal{C} \to \mathcal{C}[\mathcal{M}^{-1}]$$

satisfying the universal property that for any functor

$$F : \mathcal{C} \to \mathcal{D}$$

that sends all morphisms in $\mathcal{M}$ to invertible morphisms in $\mathcal{D}$ factors uniquely as

$$\begin{tikzcd} \mathcal{C} \arrow[r, "L"] \arrow[dr, "F"'] & \mathcal{C}[\mathcal{M}^{-1}] \arrow[d, "F'"] \\ & \mathcal{D} \end{tikzcd}$$

A special case of this construction is the localization of a category $\mathcal{C}$ by a subcategory $\mathcal{D}$. In this case the class of morphisms $\mathcal{M}$ is simply $Mor(\mathcal{D})$, the class of all morphisms in $\mathcal{D}$.

We can think of the localization (if it exists) as the category that has the same objects as $\mathcal{C}$ but in which we have defined formally an inverse for each morphism in $\mathcal{M}$. From this it is clear that the canonical localization functor is essentially surjective.

If the localization exists it is clearly unique.

Remark 6.49. We will not dwell on issues such as under what hypotheses such a localization exists, and we shall ignore any set-theoretic issues. For a clear account of the theory, see for example Kashiwara and Schapira [2006, Chap. 7]

Now we are able to define the categories of constructible and lisse $\mathbb{Q}_l$-sheaves.

Definition 6.50. Assume that X is a scheme and let l be a prime that is invertible on X, i.e. such that no residue field of a point $x \in X$ has characteristic l. The category of constructible $\mathbb{Q}_l$-sheaves on X is defined as the localization of $\mathrm{Sh}^{con}_{\mathbb{Z}_l}(X)$ by the full subcategory of torsion sheaves. The objects of this category are the same as in $\mathrm{Sh}^{con}_{\mathbb{Z}_l}(X)$ and the morphisms are given by

$$\mathrm{Hom}_{\mathbb{Q}_l}(\mathcal{F}, \mathcal{G}) = \mathrm{Hom}_{\mathbb{Z}_l}(\mathcal{F}, \mathcal{G}) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l$$

We denote the category of constructible $\mathbb{Q}_l$-sheaves on X by $\mathrm{Sh}^{con}_{\mathbb{Q}_l}(X)$. Furthermore we say that a constructible $\mathbb{Q}_l$-sheaf $\mathcal{F}$ is lisse if there exists an étale cover $\{U_i \to X\}_{i \in I}$ and lisse $\mathbb{Z}_l$-sheaves $\mathcal{F}_i$ on U_i such that

$$\mathcal{F}|_{U_i} \cong \mathcal{F}_i$$

as $\mathbb{Z}_l$-sheaves. We denote that category of lisse $\mathbb{Q}_l$-sheaves on X by $\mathrm{Sh}^{lis}_{\mathbb{Q}_l}(X)$.

Sometimes when we speak about $\mathbb{Q}_l$-sheaves we write them as $\mathcal{F} \otimes_{\mathbb{Z}_l} \mathbb{Q}_l$ where $\mathcal{F}$ is a $\mathbb{Z}_l$-sheaf. This is not an honest tensor product, merely a suggestive label.

Definition 6.51. Let X be a scheme, $\overline{x} : \operatorname{Spec}\Omega \to X$ be a geometric point in X and let l be a prime number that is invertible on X. Let $\mathcal{F}$ be a lisse $\mathbb{Q}_l$-sheaf on X. The stalk of $\mathcal{F}$ at $\overline{x}$ is defined as

$$\mathcal{F}_{\overline{x}} := (\varprojlim_n \mathcal{F}_{n,\overline{x}}) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l$$

i.e.

$$\mathcal{F}_{\overline{x}} = \mathcal{F}_{\overline{x}} \otimes_{\mathbb{Z}_l} \mathbb{Q}_l$$

where on the right hand side we view $\mathcal{F}$ as a $\mathbb{Z}_l$-sheaf.

Theorem 6.52. *We let X be a connected, locally Noetherian scheme, $\overline{x} : \operatorname{Spec}\Omega \to X$ a geometric point and let l be a prime that is invertible on X. Then the fibre functor induces an equivalence of categories*

$$\mathrm{Sh}^{lis}_{\mathbb{Q}_l}(X) \xrightarrow{\cong} \pi_1^{\text{ét}}(X, \overline{x}) - \mathrm{Rep_c}^{\mathbb{Q}_l}$$

Proof. In the same manner as in 6.46 we obtain a functor $\mathcal{F} \mapsto \mathcal{F}_{\overline{x}}$. To construct the inverse functor we let V be a a $\mathbb{Q}_l$-Representation of $\pi_1(X, \overline{x})$. By restriction V is a $\mathbb{Z}_l$-Representation, and inside V we can find a finitely genereted $\mathbb{Z}_l$-subrepresentation M such that $\pi_1(X, \overline{x})$ acts continuously on M and and we have an isomorphism $\mathbb{Q}_l[\pi_1(X, \overline{x})]$-modules

$$M \otimes_{\mathbb{Z}_l} \mathbb{Q}_l \cong V$$

By 6.46, M gives rise to a lisse $\mathbb{Z}_l$-Sheaf $\mathcal{F}_M$ and we define

$$\mathcal{F}_V := \mathcal{F}_M \otimes_{\mathbb{Z}_l} \mathbb{Q}_l$$

The choice of the $\mathbb{Z}_l$-submodule M is unique up to $\mathbb{Q}_l$-automorphisms of V and so we have that the construction of $\mathcal{F}_V$ is independent of the choice of M. We readily see that

$$\mathcal{F}_{V,\overline{x}} = V \cong M \otimes_{\mathbb{Z}_l} \mathbb{Q}_l$$

and we have thus constructed the inverse functor. $\square$

7. The pro-étale fundamental group

Throughout this chapter we will assume that our base scheme is locally Noetherian and connected. All topological groups are assumed to be Hausdorff.

7.1. **Raĭkov completions of topological groups.** We begin by recalling some facts on uniform spaces and Raĭkov completions of topological groups. For the general topological definitions we follow Bourbaki [1995].

Definition 7.1. A *filter* on a set X is a set $\mathfrak{F}$ of non-empty subsets of X satisfying the following axioms

(1) Every subset of X which contains a member of $\mathfrak{F}$ is itself a member of $\mathfrak{F}$.
(2) Finite intersections of members of $\mathfrak{F}$ are themselves members of $\mathfrak{F}$.

The standard example for us to have in mind is the following.

Example 7.2. Let X be a topological space and let $x \in X$ be a point. Then the set $\mathfrak{N}(x)$ of all neighborhoods of x is a filter. We call it the neighborhood filter of x.

We can compare filters.

Definition 7.3. Let X be a set and let $\mathfrak{F}$ and $\mathfrak{F}^{'}$ be two filters on X. We say that $\mathfrak{F}^{'}$ is *finer* than $\mathfrak{F}$, or equivalently that $\mathfrak{F}$ is coarser than $\mathfrak{F}^{'}$ if $\mathfrak{F} \subseteq \mathfrak{F}^{'}$.

This defines a partial order on the set of all filters on X.

There is a notion of a base of a filter, similar to the notion of a base of a topology on a set X.

Proposition 7.4. *Let X be a set and let $\mathfrak{B}$ be a set of subsets of X. Let $\mathfrak{F}$ be the set of all subsets of X which contain a set from $\mathfrak{B}$. Then $\mathfrak{F}$ is a filter on X if and only if $\mathfrak{B}$ satisfies the following two properties*

(1) The intersection of two sets in $\mathfrak{B}$ contains a set from $\mathfrak{B}$.
(2) $\mathfrak{B}$ is not empty and the empty set is not in $\mathfrak{B}$.

If we have $\mathfrak{B}$ and $\mathfrak{F}$ like in the above proposition, then we say that $\mathfrak{B}$ generates the filter $\mathfrak{F}$ and that $\mathfrak{B}$ is a base for $\mathfrak{F}$.

We can now define the notion of *convergence of a filter*.

Definition 7.5. Let X be a topological space, let $x \in X$ be a point and let $\mathfrak{F}$ be a filter on X. We say that $\mathfrak{F}$ converges to x, or equivalently that x is a limit point of $\mathfrak{F}$, if $\mathfrak{F}$ is finer then $\mathfrak{N}(x)$, the neighborhood filter of x.

Example 7.6. A topological space X is Hausdorff if and only if every filter $\mathfrak{F}$ on X has at most one limit point.

This approach to topology gives us a way to define continuity of functions in terms of convergence of filters, analogous to metric space theory. This suggests that we should define an analogue of Cauchy sequences and complete metric spaces. To do this we need to first define *uniform spaces* which are sets X, with a special kind of filter on the product $X \times X$.

Definition 7.7. Let X be a set. A uniform structure on X is a filter $\mathfrak{U}$ on $X \times X$ satisfying the following additional conditions

(1) Every set in $\mathfrak{U}$ contains the diagonal $\Delta = \{(x, x)|x \in X\}$.
(2) For every $V \in \mathfrak{U}$ the set

$$V^{-1} := \{(y, x) \in X \times X|(x, y) \in V\}$$

is also a member of $\mathfrak{U}$.
(3) For any $V \in \mathfrak{U}$ there is a $W \in \mathfrak{U}$ such that $W \circ W \subseteq V$, where we define

$$A \circ B := \{(x, y) \in X \times X|\ \exists z \in X \text{ such that } (x, z) \in A, (z, y) \in B\}$$

A uniform space is a pair $(X, \mathfrak{U})$ where X is a set and $\mathfrak{U}$ is a uniform structure on X. We call the members of $\mathfrak{U}$ *entourages of the uniformity on X defined by* $\mathfrak{U}$ or simply *entourages of X*.

Note that we usually just say that X is a uniform space and leave the uniform structure $\mathfrak{U}$ implicit unless we specifically have to work with it.

Given a set X with a uniform structure $\mathfrak{U}$ there is a unique way to put a topology on X that respects the uniform structure.

Proposition 7.8. *Let $(X,\mathfrak{U})$ be a uniform space. For an entourage $V \in \mathfrak{U}$ and a point $x \in X$ we denote*

$$V(x) := \{y \in X | (x,y) \in V\}$$

We then denote

$$\mathfrak{N}(x) := \{V(x) | V \in \mathfrak{U}\}$$

Then there exists a unique topology on X such that for each $x \in X$ the set $\mathfrak{N}(x)$ is the neighborhood filter of x in this topology.

Proof. Bourbaki [1995, Chap. 2.1, Prop. 1] □

We call this topology *the uniform topology on X*.

If we have a topological space X and define a uniform structure $\mathfrak{U}$ on the underlying set of X, then we say that $\mathfrak{U}$ is a *compatible uniformity* on X if the uniform topology defined by $\mathfrak{U}$ is the same as the topology on X we began with.

Definition 7.9. Let X and $X^{'}$ be uniform spaces. A mapping

$$f : X \to X^{'}$$

is said to be *uniformly continuous* if for each entourage $V^{'}$ of $X^{'}$ there exists an entourage V of X such that

$$(x,y) \in V \Rightarrow (f(x), f(y)) \in V^{'}$$

Notice that uniformly continuous maps are continuous in the uniform topology. It is however not true that every continuous map in the uniform topology is uniformly continuous.

Definition 7.10. Let X be a uniform space and let V be an entourage of X. A subset $A \subseteq X$ is said to be V-small if $A \times A \subseteq V$.

We now have all the ingredients needed to define Cauchy filters.

Definition 7.11. A filter $\mathfrak{F}$ on a uniform space X is said to be a Cauchy filter if for each entourage V of X, there is a subset $A \subseteq X$ such that A is V-small and $A \in \mathfrak{F}$.

Just like every convergent sequence in a metric space is Cauchy we have the following for filters.

Proposition 7.12. *Every convergent filter on a uniform space X is a Cauchy filter.*

Proof. Bourbaki [1995, Chap. 2.3, Prop. 2] □

It is however not true that every Cauchy filter is convergent, again completely analogous to metric space theory, which suggests the following definition.

Definition 7.13. Let X be a uniform space. We say that X is complete if every Cauchy filter on X converges.

The following theorem is the main theorem we need before we move on to applying these concepts in the setting of topological groups.

Theorem 7.14. *Let X be a uniform space. Then there exists a complete Hausdorff uniform space $\hat{X}$ and a uniformly continuous map $i : X \to \hat{X}$ such that any uniformly continuous map $f : X \to Y$, where Y is a complete Hausdorff uniform space, factors through i. That is, there exists a unique uniformly continuous map $g : \hat{X} \to Y$ such that the following diagram commutes*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "i"'] & Y \\ & \hat{X} \arrow[u, "\exists! g"'] \end{tikzcd}$$

This pair $(i, \hat{X})$ is unique up to a unique isomorphism.

Furthermore, the image $i(X)$ is dense in $\hat{X}$ and if X is Hausdorff, i is injective. That is, if X is Hausdorff, i induces an isomorphism from X onto a dense subset of $\hat{X}$.

Proof. For the first part see Bourbaki [1995, Chap. 2.3, Thm. 3]. For the second, see Bourbaki [1995, Chap. 2.3, Prop. 12] and the corollary that follows it. □

We call $\hat{X}$ the Hausdorff completion of X. Note that if X is Hausdorff and complete, then $X = \hat{X}$

Now we introduce a certain uniform structure on topological groups and the completions associated with them.

Let G be a topological group with identity e. Let $N_s(e)$ denote the collection of all symmetric open neighborhoods of e. Given any $V \in N_s(e)$ we define the following subset of $G \times G$

$$O_V := \{(x, y) \in G \times G | x^{-1}y \in V \text{ and } xy^{-1} \in V\}$$

We call a subset V of $G \times G$ symmetric if $V = V^{-1}$ where V^{-1} is as defined in the definition of uniform spaces 7.7. This is not to be confused with symmetric subsets of topological groups, such as the open subsets that belong to $N_s(e)$.

We denote by $\mathfrak{D}_G$ the family of all symmetric subsets of $G \times G$.

Definition 7.15. Let G be a topological group and define

$$\mathfrak{U}_G := \{D \in \mathfrak{D}_G | O_V \subseteq D \text{ for some } V \in N_s(e)\}$$

We call $\mathfrak{U}_G$ the *two-sided uniformity* on G.

To justify the name given in the previous definition we have the following proposition.

Proposition 7.16. *Let G be a topological group. Then $\mathfrak{U}_G$ as defined in 7.15 is a uniformity structure on G, and furthermore it is a compatible uniformity on G.*

Proof. See Arhangel'skii and Tkachenko [2008, Thm. 1.8.3] □

The following definition and proposition are very important in what follows when we define and look at properties of Noohi groups.

Definition 7.17. Let G be a topological group. The (Hausdorff) completion of G with respect to it's two-sided uniformity is called the Raĭkov completion of G and is denoted by G^*.

The canonical map $i : G \to G^*$ from 7.14 is a topological group homomorphism and so induces an isomorphism of topological groups from G to a dense subgroup of G^*. We say that G is Raĭkov complete if i is an isomorphism onto G^*.

Proposition 7.18. *Every locally compact topological group is Raĭkov complete.*

Proof. See Arhangel'skii and Tkachenko [2008, Thm. 3.6.24] □

The category of Raĭkov complete groups is closed under products and taking closed subgroups.

Proposition 7.19. *Let H be a closed topological subgroup of a Raĭkov complete group G. Then H is Raĭkov complete.*

Proof. We can embed H as a dense subgroup of H^* via i_H. Furthermore if we let j be the inclusion $j : H \to G$, then Arhangel'skii and Tkachenko [2008, Prop. 3.6.12] tells us that there exists a map j^* making the following diagram commute.

$$\begin{tikzcd} H \arrow[r, "j"] \arrow[d, "i_H"] & G \\ H^* \arrow[ur, dashed, "j^*"'] & \end{tikzcd}$$

But then $H = J(H) \subseteq j^*(H^*)$ is a dense closed subgroup, i.e.

$$H = j(H) = j^*(H^*)$$

Now the restriction $j^*|_{i_H(H)}$ induces an isomorphism

$$j^*|_{i_H(H)} : i_H(H) \to j(H) = H$$

so Arhangel'skii and Tkachenko [2008, Cor. 3.6.18] tells us that j^* induces an isomorphism

$$j^* : H^* \to j^*(H^*)$$

We have thus obtained a map

$$j^* : H^* \to H$$

which is the inverse to

$$i_H : H \to H^*$$

proving that H is Raĭkov complete. □

Proposition 7.20. *Every topological product $G = \prod_{i \in I} G_i$ of Raĭkov complete groups, is Raĭkov complete.*

Proof. See Arhangel'skii and Tkachenko [2008, 3.6.22] □

We end this section by recalling an equivalent way of defining the Raĭkov completion.

Definition 7.21. A family $\mathfrak{F}$ of subsets of a topological space X with topology $\mathcal{T}$ is called *an open filter* if there exists a filter $\mathfrak{G}$ on X such that

$$\mathfrak{F} = \mathfrak{G} \cap \mathcal{T}$$

An *open filter base* is a filter base all of whose members are open sets.

Definition 7.22. Let G be a topological group. An open filter $\mathfrak{F}$ on G is called *shrinking* if for any $B \in \mathfrak{F}$ there exists an $A \in \mathfrak{F}$ and open neighborhoods U and V of 1 such that

$$UAV \subseteq B$$

An open filter on G is called *canonical* if it is both Cauchy and shrinking.

Now Arhangel'skii and Tkachenko [2008, Thm. 3.6.25] gives us a description of the Raĭkov completion G^* as having elements all canonical filters on G with the group operation given by

$$\mathfrak{F}_1 \circ \mathfrak{F}_2 = o([\mathfrak{F}_1\mathfrak{F}_2])$$

where

(1) $[\mathfrak{F}_1\mathfrak{F}_2] := \{AB | A \in \mathfrak{F}_1 \text{ and } B \in \mathfrak{F}_2\}$ for any two families $\mathfrak{F}_1$ and $\mathfrak{F}_2$ of subsets of G, and

(2) $o(\mathfrak{F}) = \{U \subseteq G | U \text{ is open in } G \text{ and } \exists V \in \mathfrak{F} \text{ such that } V \subseteq U\}$ for any family $\mathfrak{F}$ of subsets of G.

The inverse of $\mathfrak{F} \in G^*$ is given by

$$\mathfrak{F}^{-1} := \{U^{-1} | U \in \mathfrak{F}\}$$

where $U^{-1} := \{a^{-1} | a \in U\}$.

The identity of G^* is

$$B_1 := \{U | U \subseteq G \text{ is an open neighborhood of the identity } 1\}$$

and the canonical inclusion $i : G \to G^*$ sends $x \in G$ to B_x, the canonical filter of all open neighborhoods of x.

Finally we put a topology on G^* by setting for each open $U \subset G$,

$$U^* := \{\mathfrak{F} \in G^* | U \in \mathfrak{F}\}$$

and obtaining a base for a topology

$$\mathcal{B} := \{U^* | U \in \mathcal{T}\}$$

where $\mathcal{T}$ is the topology on G.

We refer to Arhangel'skii and Tkachenko [2008, Chap. 3.6] for proofs that these operations and notions are well defined and endow G^* with the structure of a topological group isomorphic to the Raĭkov completion of G.

7.2. **Noohi groups and infinite Galois categories.** Let G be a topological group, and consider the forgetful functor

$$F_G : G - \mathbf{Set} \to \mathbf{Set}$$

where $G - \mathbf{Set}$ is the category of discrete sets with a continous action of G.

The automorphism group of this functor, $\mathrm{Aut}(F_G)$ is given the structure of a topological group by using the compact-open topology on $\mathrm{Aut}(F_G(X))$ for each G-set X and viewing $\mathrm{Aut}(F_G)$ as the inverse limit of those.

Then there is a canonical group homomorphism

$$G \to \mathrm{Aut}(F_G)$$

The following definition was made in Noohi [2008]. In there the author calls these groups *prodiscrete groups*. The term *Noohi group* was introduced in Bhatt and Scholze [2015] to avoid confusion, as Noohi groups are not pro-(discrete groups).

Definition 7.23. A topological group G is called a *Noohi group* if the canonical map $G \to \mathrm{Aut}(F_G)$ induces an isomorphism of topological groups

$$G \cong \mathrm{Aut}(F_G)$$

This is a natural generalization of profinite groups. We saw the following in 4.

Proposition 7.24. *Let G be a topological group and consider the forgetful functor*

$$F : G - \mathrm{Rep}_{\mathrm{c}} \to \boldsymbol{FinSet}$$

The canonical map $G \to \mathrm{Aut}(F)$ induces an isomorphism of topological groups

$$G \cong \mathrm{Aut}(F)$$

if and only if G is profinite.

To provide examples we look at another characterization of Noohi groups.

Let us first look at a lemma.

Lemma 7.25. *For any set S with the discrete topology, the group $\mathrm{Aut}(S)$ is Raĭkov complete for the compact-open topology.*

Proof. This is Bhatt and Scholze [2015, Lem. 7.1.4]. □

Proposition 7.26. *Let G be be a topological group with a basis of open neighborhoods of $1 \in G$ given by open subgroups. Then there is a natural isomorphism $\mathrm{Aut}(F_G) \cong G^*$. In particular G is Noohi if and only if it is Raĭkov complete.*

Proof. See Bhatt and Scholze [2015, Prop. 7.1.5]. □

Corollary 7.27. *Let S be a discrete set. Then $\mathrm{Aut}(S)$ is a Noohi group.*

Proof. Let $B(K, U)$ be an open neighborhood of 1 consisting of all those maps that send a compact K into an open U. These sets form a filter base for the neighborhood filter of 1 on $\mathrm{Aut}(S)$. Now since $B(K, U)$ is a neighborhood of 1 then $K = id_S(K) \subseteq U$. Therefore

$$B(K, K) \subseteq B(K, U)$$

but $B(K, K)$ is an open subgroup of $\mathrm{Aut}(S)$. So we have shown that $\mathrm{Aut}(S)$ has a basis of open neighborhoods of 1 consisting of open subgroups and by 7.25 it is Raĭkov complete. ☐

Corollary 7.28. *Let H be a Noohi group and G a closed topological subgroup of G. Then G is Noohi.*

Proof. Clearly G has a basis of open neighborhoods of 1 consisting of open subgroups. Furthermore G is Raĭkov complete by 7.19. ☐

Corollary 7.29. *If G_i is Noohi for each $i \in I$, then $G := \prod_{i\in I} G_i$ is Noohi.*

Proof. G has a basis of open neighborhoods of 1 given by open subgroups since each of the G_i does. And by 7.20 G is Raĭkov complete. ☐

This characterization allows us to generate a wealth of examples.

Corollary 7.30. *Any locally compact group G with a basis of open neighborhoods of 1 given by open subgroups is Noohi.*

Proof. Locally compact groups are Raĭkov complete by 7.18. ☐

Example 7.31. Any profinite group and any discrete group is Noohi.

The most important example for us is the following.

Example 7.32. Fix l a prime. Then the local field $\mathbb{Q}_l$ is locally compact and thus Noohi by 7.30. Furthermore $GL_n(\mathbb{Q}_l)$ is locally compact and it has a basis of open neighborhoods of 1 consisting of open subgroups, so by 7.30 we see that

$$GL_n(\mathbb{Q}_l) \text{ is a Noohi group.}$$

We even have a stronger statement than 7.32, namely in Bhatt and Scholze [2015, Ex. 7.1.7] the following example is given.

Example 7.33. Let E be an algebraic extension of $\mathbb{Q}_l$, then the group $GL_n(E)$ is Noohi under the colimit topology, i.e. the topology induced by expressing E as a union of finite extensions. In particular, $GL_n(\bar{\mathbb{Q}}_l)$ is Noohi.

In chapter 3 we saw how Galois categories corresponded naturally to profinite groups, i.e. that each profinite group gives rise to a Galois category and to each Galois category corresponds a profinite group (namely the fundamental group of the category), and that this correspondence is functorial.

Here we are going to introduce a generalization of the Galois categories, called infinite Galois gategories, and show how they correspond naturally to Noohi groups.

Remark 7.34. In what follows we shall use the term "infinite Galois categories" for what Bhatt and Scholze term "tame infinite Galois categories". They define a more general version of infinite Galois categories, but the tame ones are precisely those that correspond to Noohi groups.

Definition 7.35. Let $\mathcal{C}$ be a category and $F : \mathcal{C} \to \mathbf{Set}$ a functor. Then we say that $\mathcal{C}$ is an infinite Galois category with a fibre functor F and fundamental group $\pi_1(\mathcal{C}, F) := \mathrm{Aut}(F)$ if the following conditions are satisfied.

(1) $\mathcal{C}$ admits all colimits and all finite limits.
(2) Each objcect $X \in \mathcal{C}$ is isomorphic to a coproduct of connected objects.
(3) $\mathcal{C}$ is generated under colimits by a set of connected objects.
(4) F is faithful and commutes with colimits and finite limits.
(5) If $f : Y \to X$ is a morphism in $\mathcal{C}$ such that $F(f) : F(Y) \to F(X)$ is a bijection (i.e. an isomorphism in $\mathbf{Set}$) then f is an isomorphism.
(6) For any connected $X \in \mathcal{C}$ the fundamental group $\pi_1(\mathcal{C}, F)$ acts transitively on the fibre $F(X)$.

Remarks 7.36. (1) A functor satisfying condition 5 is called *conservative*.
(2) Condition 6 is not in the definition given by Bhatt-Scholze. It is the extra condition that elevates what they call an infinite Galois category to what they call a tame infinite Galois category. Since we are only concerned with those infinite Galois categories that are tame, we include the tameness condition in our definition.
(3) Notice that condition 3 contains the set-theoretic restrictions on $\mathcal{C}$ that we have a *set* and not any class of connected generators. In these thesis we mostly ignore these set-theoretic matters.

Proposition 7.37. *Let G be a Noohi group. Then $G - \boldsymbol{Set}$ is an infinite Galois category with a fibre functor $F_G : G - \boldsymbol{Set} \to \boldsymbol{Set}$*

Proof. We go through the conditions in 7.35.

(1) It is easy to describe limits and colimits in $G - \mathbf{Set}$, namely they are simply the limits and colimits in $\mathbf{Set}$ with the added structure of a continuous G-action. Let $\Psi : \mathcal{I} \to G - \mathbf{Set}$ be a functor from some category $\mathcal{I}$. Then we can describe the colimit

of Ψ as

$$\operatorname{colim}\Psi = \coprod_{i\in\mathcal{I}} \Psi(i)/\sim$$

where the equivalence relation $\sim$ is given by

$$x_i \sim x_j \Leftrightarrow \exists f : i \to j \text{ with } \Psi(f)(x_i) = x_j$$

where $x_i \in \Psi(i)$ and $x_j \in \Psi(j)$. We have the obvious inclusion into the disjoint union, and the only thing we need to show is that $\sim$ respects the natural G-action on $\coprod_{i\in\mathcal{I}} \Psi(i)$. Assume $x_i \sim x_j$ where $x_i \in \Psi(i)$ and $x_j \in \Psi(j)$. Then there exists some map $f : i \to j$ in $\mathcal{I}$ such that $\Psi(f)(x_i) = x_j$. But then

$$\Psi(f)(gx_i) = g\Psi(x_i) = gx_j$$

Similarly we can describe the limit of $\Phi : \mathcal{I} \to \mathbf{Set}$ as

$$\lim \Phi = \{(x_i) \in \prod_{i\in\mathcal{I}} \Phi(i) | \forall f : i \to j, \Phi(f)(x_i) = x_j\}$$

This carries over to $G - \mathbf{Set}$ with the one caveat that $\mathcal{I}$ has to be a finite category, because the product $\prod_{i\in I} X_i$ of discrete spaces is discrete if and only if I is finite.

Thus $G - \mathbf{Set}$ admits all colimits and all *finite* limits.

(2) The connected objects in $G - \mathbf{Set}$ are precisely the sets on which G acts transitively. The coproduct is the disjoint union and so we can write any $X \in G - \mathbf{Set}$ as a coproduct of connected objects simply by decomposing it into the disjoint union of the orbits of the G-action.

(3) What remains to be seen here is that the class of all connected objects in $G - \mathrm{Rep}_{\mathrm{c}}$ is a *set* and not a proper class.

Now notice that if X and Y are discrete sets with a transitive G-action, i.e. connected objects in $G - \mathbf{Set}$, and if there exist $x \in X$ and $y \in Y$ such that the stabilizers G_x and G_y agree, then we have natural bijections by the orbit-stabilizer theorem

$$G/G_x \to X$$
$$h \cdot G_x \mapsto hx$$

and

$$G/G_y \to Y$$
$$h \cdot G_y \mapsto hy$$

so we can define a bijective map $\phi : X \to Y$ as the composition

$$hx \mapsto h \cdot G_x = h \cdot G_y \mapsto hy$$

clearly ϕ is a G-map and so X and Y are isomorphic as G-sets.

If we on the other hand assume X and Y are isomorphic as G-sets and choose some $x \in X$ and let y be the image of x under this isomorphism, then $G_x = G_y$.

We have therefore established a bijection between a certain class of subgroups of G and the class of isomorphism classes of connected objects in $G - \mathbf{Set}$. Therefore the cardinality of the connected objects, up to isomorphisms, is less then or equal to the cardinality of the power set of G which is a set and not a proper class.

(4) This is obvious by construction.

(5) Again this is obvious by construction since the isomorphisms in $G - \mathbf{Set}$ are precisely the G-maps that are bijections.

(6) This is again obvious since the action of $\pi_1(G - \mathbf{Set}, F_G) = \mathrm{Aut}(F_G) = G$ on connected objects is by definition transitive.

□

We now state and prove the main theorem on infinite Galois categories. It is formally very close to the main theorem on (finite) Galois categories, where now infinite Galois categories take the place of (finite) Galois categories, and Noohi groups take the place of profinite groups. This is Bhatt and Scholze [2015, Thm. 7.2.5]

Theorem 7.38. *Let $\mathcal{C}$ be an infinite Galois category with a fibre functor F. Then we have the following.*

(1) $\pi_1(\mathcal{C}, F)$ is a Noohi group.

(2) F induces an equivalence of categories

$$\mathcal{C} \cong \pi_1(\mathcal{C}, F) - \boldsymbol{Set}$$

Proof. (1) We fix a set $\{X_i\}_{i\in I}$ of connected generators of $\mathcal{C}$. Like we saw in the proof of 7.26 we can view $\pi_1(\mathcal{C}, F)$ as a closed subgroup of $\prod_{i\in I} \mathrm{Aut}(F(X_i))$.

Now each $\mathrm{Aut}(F(X_i))$ is Noohi by 7.27 so the product $\prod_{i\in I} \mathrm{Aut}(F(X_i))$ is Noohi by 7.29 Therefore $\pi_1(\mathcal{C}, F)$ is a closed subgroup of a Noohi group, and thus Noohi by 7.28.

(2) By condition 6, the tameness condition, in the definition of infinite Galois categories, $\pi_1(\mathcal{C}, F)$ acts transitively on the fibre $F(X)$ of any connected object $X \in \mathcal{C}$. Therefore, if X is any object in $\mathcal{C}$ and we write

$$X = \coprod X_i$$

for some connected X_i's, we obtain, since F commutes with colimits

$$F(X) = \coprod F(X_i)$$

where $\pi_1(\mathcal{C}, F)$ acts transitively on each $F(X_i)$. The connected objects in $\pi_1(\mathcal{C}, F) - \mathbf{Set}$ are, as mentioned before, precisely the objects on which $\pi_1(\mathcal{C}, F)$ acts transitively so the functor

$$F_c : \mathcal{C} \to \pi_1(\mathcal{C}, F) - \mathbf{Set}$$

through which F factors,

$$F = F_{\pi_1(\mathcal{C},F)} \circ F_c$$

where $F_{\pi_1(\mathcal{C},F)} : \pi_1(\mathcal{C}, F) - \mathbf{Set} \to \mathbf{Set}$ is the forgetful functor, preserves connected components.

By assumption, the fibre functor F is faithful and so F_c is also faithful.

To check fullness we let $X, Y \in \mathcal{C}$ be given and consider a map in $\pi_1(\mathcal{C}, F) - \mathbf{Set}$:

$$g : F_c(X) \to F_c(Y)$$

Then the graph Γ_g of g can be interpreted as the fiber product

$$\begin{tikzcd} \Gamma_g \arrow[r] \arrow[d] & F_c(Y) \arrow[d, "id_{F_c(Y)}"] \\ F_c(X) \arrow[r, "g"'] & F_c(Y) \end{tikzcd}$$

The fibre product is a finite limit with which F commutes, so there exists some map $f : X \to Y$ in $\mathcal{C}$ and an object $\Gamma \in \mathcal{C}$ such that $F(\Gamma) = F_{\pi_1(\mathcal{C},F)}(\Gamma_g)$ and $F(f) = F_{\pi_1(\mathcal{C},F)}(g)$ and we have the following diagram

$$\begin{tikzcd} \Gamma \arrow[r] \arrow[d] & Y \arrow[d, "id_Y"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

Thus F is fully faithful and the forgetful functor $F_{\pi_1(\mathcal{C},F)}$ is fully faithful, so F_c is fully faithful.

To conclude the proof that F_c enduces an equivalence of categories, we must check that it is essentially surjective. Since we have shown that F_c preserves connected components, we just have to check that for every connected $S \in \pi_1(\mathcal{C}, F) - \mathbf{Set}$ there exists some connected $X \in \mathcal{C}$ such that $F_c(X) = S$.

So let a connected $S \in \pi_1(\mathcal{C}, F) - \mathbf{Set}$ be given. Then as we have seen before, $\pi_1(\mathcal{C}, F)$ acts transitively on S and there exists an open subgroup $U \leq \pi_1(\mathcal{C}, F)$ (namely $U = \mathrm{Stab}_s$ for some $s \in S$) such that as $\pi_1(\mathcal{C}, F)$-spaces

$$\pi_1(\mathcal{C}, F)/U \cong S$$

We can as before view $\pi_1(\mathcal{C}, F)$ as a closed subgroup of $\prod_{i \in I} \mathrm{Aut}(F(X_i))$. Therefore there exist finitely many connected generators $X_{i_1}, \ldots, X_{i_n}$ and for each j an element $x_{i_j} \in F_c(X_{i_j})$ such that U contains the open subgroup $U^{'}$ of $\pi_1(\mathcal{C}, F)$ that fixes all of the x_{i_j}. By construction we see that $\pi_1(\mathcal{C}, F)/U^{'}$ is (isomorphic to) the connected component of $F_c(X_{i_1}) \times \cdots \times F_c(X_{i_n})$ containing $x := (x_{i_j})_{j=1}^n$. Thus, if we let $X_{U'}$ be the connected component of $X_{i_1} \times \cdots \times X_{i_n}$ we see that

$$F_c(X_{U'}) = \pi_1(\mathcal{C}, F)/U^{'}$$

Now U defines a congruence on $X_{U'}$ via the transitive action of $\pi_1(\mathcal{C}, F)$ on $F(X_{U'})$ and since colimits exist in $\mathcal{C}$ we have a quotient object $X_U := X_{U'}/U$. Since F_c commutes with colimits, and since the congruence on $X_{U'}$ is exactly such that the corresponding congruence on the image $F_c(X_U^{'})$ induces the usual quotient in $\pi_1(\mathcal{C}, F) - \mathbf{Set}$ we have

$$F_c(X_U) = (\pi_1(\mathcal{C}, F)/U^{'})/U \cong \pi_1(\mathcal{C}, F)/U \cong S$$

$\square$

7.3. **The pro-étale site and pro-étale fundamental groups.** We now use the theory of infinite Galois categories to define the pro-étale fundamental group of a scheme X and relate that to the étale fundamental group. In order to do so we must first define the pro-étale site $X_{\text{proét}}$.

Definition 7.39. A map of schemes $f : Y \to X$ is said to be weakly étale if it is flat, and the diagonal

$$\Delta_{Y/X} : Y \to Y \times_X Y$$

is flat as well.

Remark 7.40. Every étale map is weakly étale.

If the structure map $Y \to X$ of an X-scheme is weakly étale, we say that Y is *a weakly étale X-scheme.*

It is clear that the property of being weakly étale is stable under base chance and composition, and that all X-morphisms of weakly étale X-schemes are weakly étale.

Lemma 7.41. *We let $f : Y \to X$ be a weakly étale morphism of schemes.*

(1) (Composition) *If $g : Z \to Y$ is another weakly étale morphism, then the composition $f \circ g : Z \to X$ is weakly étale.*
(2) (Base change) *If $g : X^{'} \to X$ is any morphism, then the base change map*

$$p_2 : Y^{'} = Y \times_X X^{'} \to X^{'}$$

is a weakly étale morphism.

Proof. (1) The composition $f \circ g$ is flat by 5.36. For the flatness of the diagonal $Z \to Z \times_X Z$ we notice that we can write

$$Z \times_Y Z = (Z \times_X Z) \times_{Y \times_X Y} Y$$

and furthermore that the canonical map

$$Z \times_Y Z \to Z \times_X Z$$

from the universal property for the fiber product $Z \times_X Z$ coincides with the first projection

$$p_1 : (Z \times_X Z) \times_{Y \times_X Y} Y \to Z \times_X Z$$

and so the diagonal

$$Z \to Z \times_X Z$$

is the composition

$$Z \to Z \times_Y Z = (Z \times_X Z) \times_{Y \times_X Y} Y \to Z \times_X Z$$

By 5.36 the base change of a flat map is flat, and since by assumption the diagonal $Y \to Y \times_X Y$ is flat, we deduce that $p_1 : (Z \times_X Z) \times_{Y \times_X Y} Y \to Z \times_X Z$ is flat. Also by assumption, $Z \to Z \times_Y Z$ is flat and again by 5.36 we have that the composition $Z \to Z \times_X Z$ is flat and hence $f \circ g$ is weakly étale.

(2) By 5.36 the base change $Y^{'} \to X^{'}$ is flat. We then only have to show that the diagonal

$$Y^{'} \to Y^{'} \times_{X^{'}} Y^{'}$$

is flat. But notice that this map is the base change of the flat (by assumption) map $Y \to Y \times_X Y$ by the map $g : X^{'} \to X$ and so is flat by 5.36.

$\square$

Lemma 7.42. *Let $Y \to X$ and $Z \to X$ be weakly étale X-schemes. Then any morphism $f : Y \to Z$ over X is weakly étale.*

Proof. We can write $f : Y \to Z$ as the composition

$$Y \to Y \times_X Z \to Z$$

The second morphism is flat by 5.36 since it is the base change of the flat morphism $Y \to X$ by the morphism $Z \to X$. The first morphism is the base change of the flat morphism $Z \to Z \times_X Z$ by the morphism $Y \times_X Z \to Z \times_X Z$ and thus again flat. The composition is then flat.

Now the morphism $Y \times_Z Y \to Y \times_X Y$ is an immersion. Thus Stacks Project [2017, 094R] implies that since $Y \to Y \times_X Y$ is flat by assumption, we can conclude that $Y \to Y \times_Z Y$ is flat. □

Given a scheme X we construct the *pro-étale site on* X.

Definition 7.43. Let X be a scheme. We define the *pro-étale site on* X, denoted by $X_{\text{proét}}$, by letting the underlying category be the full subcategory of **Sch/X** consisting of weakly-étale schemes over X and declare fpqc coverings to be coverings.

The category $X_{\text{proét}}$ behaves rather nicely under taking limits. This is Bhatt and Scholze [2015, Lem. 4.1.8]

Lemma 7.44. *The category $X_{\text{proét}}$ is closed under finite limits, and the full subcategory spanned by all affine weakly-étale maps $Y \to X$ is closed under all small limits. Furthermore, all these limits agree with the limits in* ***Sch/X***.

Proof. Categories that have final objects and all fibre products have finite limits. So for the first part it suffices for us to show that $X_{\text{proét}}$ has a final object and all pullbacks.

It is clear that any $Y \in X_{\text{proét}}$ has a unique morphism $Y \to X$, namely the structure morphism. Therefore $X \in X_{\text{proét}}$ is the final object.

Now assume we have maps

$$\begin{tikzcd} & Y_2 \arrow[d] \\ Y_1 \arrow[r] & Y \end{tikzcd}$$

in $X_{\text{proét}}$. The fibre product $Y_1 \times_Y Y_2$ exists in the category of schemes and furthermore by 7.42, the maps $Y_i \to Y$, for $i = 1, 2$ are weakly étale and so by 7.41 the composition

$$Y_1 \times_Y Y_2 \to Y_1 \to X$$

is weakly étale. Thus the fibre product is in $X_{\text{proét}}$.

The same kind of argument coupled with the fact that the fibre product of two affine schemes over a third affine scheme is also affine, shows that the full subcategory of $X_{\text{proét}}$ consisting of affine weakly étale X-schemes is closed under all finite limits. To show that this subcategory is complete, i.e. is closed under all limits, it is then enough to show that it is closed under all cofiltered limits.

Flatness is local on the base so we can base-change to an affine open subscheme $\operatorname{Spec}(A) \to X$. Now a cofiltered limit of affine $\operatorname{Spec}(A)$-schemes $\{\operatorname{Spec}(B_i)\}_{i\in\mathcal{I}}$ is an affine $\operatorname{Spec}(A)$-scheme $\operatorname{Spec}(B)$ and corresponds to a filtered colimit of flat A-algebras $\{B_i\}_{i\in\mathcal{I}}$. Filtered colimits preserve flatness so both $A \to B$ and $B \to B \otimes_A B$ are flat and $\operatorname{Spec}(B) \to \operatorname{Spec}(A)$ is weakly étale. $\square$

Definition 7.45. Let X be a scheme and $U \in X_{\text{proét}}$. Then U is called *pro-étale affine* if we can write

$$U = \lim_i U_i$$

for a small cofiltered diagram $i \mapsto U_i$ where each U_i is an affine scheme in $X_{\text{proét}}$. The full subcategory of $X_{\text{proét}}$ spanned by pro-étale affines is denoted by $X_{\text{proét}}^{aff}$.

The following lemma, Bhatt and Scholze [2015, Lem. 4.2.4], tells us that in some sense these pro-étale affine $U \in X_{\text{proét}}$ govern the behavior of all of $X_{\text{proét}}$

Lemma 7.46. *The topos* $\operatorname{Sh}(X_{\text{proét}})$ *is generated by* $X_{\text{proét}}^{aff}$*, i.e. for each* $Y \in X_{\text{proét}}$ *there is a family* $U_i \in X_{\text{proét}}^{aff}$ *such that* Y *admits a surjection*

$$\coprod_i U_i \to Y$$

We have a similar proposition about pro-étale sheaves as 6.30 about étale sheaves, namely that it suffices to check the sheaf property on very specific classes of coverings. This is Bhatt and Scholze [2015, Lem. 4.2.6]

Proposition 7.47. *Let* $\mathcal{F}$ *be a presheaf (of sets, or Abelian groups etc.) on* $X_{\text{proét}}$*. Then* $\mathcal{F}$ *is a sheaf if and only if the sheaf property holds for*

(1) A surjective $V \to U$ *in* $X_{\text{proét}}^{aff}$
(2) Any Zariski covering $\{U_i \to U\}$

Étale maps of schemes are weakly-étale and étale coverings are fpqc so we obtain a morphism of topoi

$$\nu : \mathrm{Sh}(X_{\text{proét}}) \to \mathrm{Sh}(X_{\text{ét}})$$

We want to consider the pushforward $\nu_* : \mathrm{Sh}(X_{\text{proét}}) \to \mathrm{Sh}(X_{\text{ét}})$ and the pullback $\nu^* : \mathrm{Sh}(X_{\text{ét}}) \to \mathrm{Sh}(X_{\text{proét}})$. It is easier to describe the pushforward; it is simply the restriction in the following manner. Let $U \in X_{\text{ét}}$ and let $F\mathrm{Sh}(X_{\text{proét}})$ then

$$\nu_*\mathcal{F}(U) = \mathcal{F}(U)$$

Let us then look at the pullback.

The following is Bhatt and Scholze [2015, Lem. 5.1.1]

Lemma 7.48. *Let $\mathcal{F} \in \mathrm{Sh}(X_{\text{ét}})$ and $U \in X^{aff}_{\text{proét}}$ with a presentation $U = \lim_i U_i$. Then we have*

$$\nu^*\mathcal{F}(U) = \mathrm{colim}_i \mathcal{F}(U_i)$$

Proof. The proof proceeds by reducing to the affine case $X = \mathrm{Spec}\,(A)$ and using results on ind-étale algebras. We refer to Bhatt and Scholze [2015] for details. $\square$

As a first consequence of the above lemma and description of the pushforward, we obtain the following. This is [Bhatt and Scholze, 2015, Lem. 5.1.2]

Lemma 7.49. *The pullback $\nu^* : Sh(X_{\text{ét}}) \to \mathrm{Sh}(X_{\text{proét}})$ is fully faithful. Its essential image consists exactly of those sheaves $\mathcal{F}$ with $\mathcal{F}(U) = \mathrm{colim}_i\mathcal{F}(U_i)$ for any $U \in X^{aff}_{\text{proét}}$ with presentation $U = \lim_i U_i$.*

Proof. A sheaf $\mathcal{F} \in \mathrm{Sh}(X_{\text{ét}})$ is uniquely determined by its values on the full subcategory of $X_{\text{ét}}$ consisting of affine schemes étale over X. So 7.48 along with our description of the pushforward imply that

$$\mathcal{F}(U) = \nu_*\nu^*\mathcal{F}(U)$$

for any affine $U \in X_{\text{ét}}$ and thus

$$\mathcal{F} \cong \nu_*\nu^*\mathcal{F}$$

The full-faithfulness of ν^* now follows formally from general results in category theory.

Now assume we have a sheaf $\mathcal{G} \in \mathrm{Sh}(X_{\text{proét}})$ such that for each $U \in X^{aff}_{\text{proét}}$ with presentation $U = \lim_i U_i$ we have $\mathcal{G}(U) = \mathrm{colim}_i U_i$.

Then from 7.48 and 7.46 we obtain that

$$\nu^*\nu_*\mathcal{G} \to \mathcal{G}$$

is an isomorphism, so $\mathcal{G}$ lies in the essential image. The other inclusion is clear from construction. $\square$

This essential image is of interest to us, so we give the sheaves that lie in it a name.

Definition 7.50. A sheaf $\mathcal{F} \in \mathrm{Sh}(X_{\text{proét}})$ is called *classical* if it lies in the essential image of ν^*.

So a sheaf is classical if and only if $\nu^*\nu_*\mathcal{F} \to \mathcal{F}$ is an isomorphism.

The following lemma gives us another way to recognize classical sheaves.

Lemma 7.51. *Let $\mathcal{F}$ be a sheaf on $X_{\text{proét}}$. Assume we have some pro-étale cover $\{Y_i \to X\}$ such that $\mathcal{F}|_{Y_i}$ is classical. Then $\mathcal{F}$ itself is classical.*

Proof. See Bhatt and Scholze [2015, Lem. 5.1.4] $\square$

We now define three important classes of sheaves on $X_{\text{proét}}$.

Definition 7.52. Let X be a locally Noetherian scheme. Let $\mathcal{F} \in \mathrm{Sh}(X_{\text{proét}})$ be given. We say that

(1) $\mathcal{F}$ is locally constant if there exist a covering $\{X_i \to X\}$ in $X_{\text{proét}}$ such that $\mathcal{F}|_{X_i}$ is constant for all i.
(2) $\mathcal{F}$ is locally weakly constant if there exists a covering $\{Y_i \to X\}$ in $X_{\text{proét}}$ such that each Y_i is qcqs and such that $\mathcal{F}|_{Y_i}$ is the pullback of a classical sheaf on the profinite set $\pi_0(Y_i)$.
(3) $\mathcal{F}$ is a geometric covering if there exists an étale X-scheme, satisfying the valuative criterion of properness, that represents $\mathcal{F}$.

We use the notation $\mathrm{Loc}(X_{\text{proét}}), w\mathrm{Loc}(X_{\text{proét}})$ and $\mathrm{Cov}(X_{\text{proét}})$ for the corresponding full subcategories of $\mathrm{Sh}(X_{\text{proét}})$.

The following is Bhatt and Scholze [2015, Ex. 7.3.5]

Example 7.53. If $X = \mathrm{Spec}\,(k)$ is the spectrum of a field, then

$$\mathrm{Loc}(X_{\text{proét}}) = w\mathrm{Loc}(X_{\text{proét}}) = \mathrm{Cov}(X_{\text{proét}}) = \mathrm{Sh}(X_{\text{ét}})$$

Furthermore the same holds for any finite scheme of Krull dimension 0.

These three subcategories of $\mathrm{Sh}(X_{\text{proét}})$ actually coincide in general and so give us equivalent ways to characterise local systems.

Proposition 7.54. *As subcategories of* $\mathrm{Sh}(X_{\text{proét}})$ *we have*

$$\mathrm{Loc}(X_{\text{proét}}) \cong w\mathrm{Loc}(X_{\text{proét}}) \cong \mathrm{Cov}(X_{\text{proét}})$$

Proof. The property of a pro-étale sheaf $\mathcal{F}$ belonging to any of the subcategories $\mathrm{Loc}(X_{\text{proét}})$, $w\mathrm{Loc}(X_{\text{proét}})$ or $\mathrm{Cov}(X_{\text{proét}})$ is local in the Zariski topology on X, so we can reduce to the case where X is Noetherian.

It is clear that $\mathrm{Loc}(X_{\text{proét}}) \subseteq w\mathrm{Loc}(X_{\text{proét}})$. Next we want to show that any locally weakly constant sheaf is a geometric covering. So fix some $\mathcal{F} \in w\mathrm{Loc}(X_{\text{proét}})$, and let $\{Y_i \to X\}$ be a covering in $X_{\text{proét}}$ such that each Y_i is qcqs and such that each $\mathcal{F}|_{Y_i}$ is a pullback of a classical sheaf on $\pi_0(Y_i)$. Then by Bhatt and Scholze [2015, Lem. 7.3.6] we know the following holds for each $\mathcal{F}_i := \mathcal{F}|_{Y_i}$.

(1) $\mathcal{F}_i$ is representable by an algebraic space that is étale over Y_i.
(2) $\mathcal{F}_i$ satisfies the valuative criterion of properness.
(3) The diagonal $\Delta : \mathcal{F}_i \to \mathcal{F}_i \times_{Y_i} \mathcal{F}_i$ is a filtered colimit of clopen immersions.

Fpqc descent then gives us that these three properties also hold for $\mathcal{F}$. If we can show that $\mathcal{F}$ is not only an algebraic space but a scheme then we have shown that $\mathcal{F}$ is a geometric covering.

By Stacks Project [2017, Tag 03XX] $\mathcal{F}$ is representable if it is locally quasi-finite and seperated. But since $\mathcal{F}$ is étale over X it is locally quasi-finite (see Stacks Project [2017, Tag 03WS]). First we show that the diagonal

$$\mathcal{F} \to \mathcal{F} \times_X \mathcal{F}$$

is quasi-compact, i.e. that $\mathcal{F}$ is quasi-seperated over X. Let U be an affine open subset of $\mathcal{F} \times_X \mathcal{F}$, we want to show that the pre-image of U is quasi-compact as a topological space. Notice that we can view the pre-image as an open subset of U since the diagonal morphism is a homeomorphism onto its image, and the image is open. Now since $U \to \mathcal{F} \times_X \mathcal{F}$ is an open immersion, it is étale 5.43, and thus the composition $U \to \mathcal{F} \times_X \mathcal{F} \to \mathcal{F} \to X$ is étale. But then U is Noetherian since X is Noetherian. Every open subspace of a notherian topological space is quasi-compact, which proves the claim.

Now we want to show that the diagonal is actually closed in $\mathcal{F} \times_X \mathcal{F}$. To do that it is enough to show that it's closed in any open set in some open covering of $\mathcal{F} \times_X \mathcal{F}$. There exists an open covering of $\mathcal{F} \times_X \mathcal{F}$ by quasi-compact open sets (this is of course true for schemes, and since for any algebraic space S there exists a scheme T and an étale surjection $T \to S$, this follows).

So now let U be a quasi-compact open subset of $\mathcal{F}\times_X\mathcal{F}$ and consider the intersection $\Delta\cap U$, where Δ is the diagonal. The diagonal morphism is a cofiltered limit of clopen immersions as we saw above, and therefore we can write

$$\Delta \cap U = \bigcup V_i$$

where the V_i's are clopen. But $\Delta \cap U$ is quasi-compact, so we may assume the union is finite, i.e. $\Delta\cap U$ is a finite union of closed subsets, and hence closed. Therefore $\mathcal{F} \to X$ is seperated, and $\mathcal{F}$ is a scheme.

Let us now prove that $\mathrm{Cov}(X_{\text{proét}}) \subseteq w\mathrm{Loc}(X_{\text{proét}})$. Let $F \in \mathrm{Cov}(X_{\text{proét}})$ be given. For any qcqs $U \in X_{\text{ét}}$ and a map $\phi : U \to F$ we can factor it as

$$U \to L \to F$$

where L is finite locally constant. This implies that for any w-contractible $Y \in X_{\text{proét}}$ the restriction $F|_Y$ is a filtered colimit of finite locally constant sheaves. This shows that F is locally weakly constant.

Finally we want to show that $w\mathrm{Loc}(X_{\text{proét}}) \subset \mathrm{Loc}(X_{\text{proét}})$. Let $\mathcal{F} \in w\mathrm{Loc}(X_{\text{proét}})$. There exists a qcqs w-contractible cover $f : Y \to X$ such that the restriction of $\mathcal{F}$ to Y is a pullback of a classical on the profinite set $\pi_0(Y)$, i.e.

$$\mathcal{F}|_Y = \pi^*\mathcal{G}$$

where $\pi : Y \to \pi_0(Y)$ is the natural map and $\mathcal{G} \in \mathrm{Sh}(\pi_0(Y)_{\text{ét}})$. If we can show that $\mathcal{G}$ is locally constant, we are done.

Since X is Noetherian, it has a finite number of generic points. Let X_η be the collection of those, and let $Y_\eta \subset Y$ be the fibre over X_η, and denote the inclusion by $g : Y_\eta \to Y$. Take a qcqs w-contractible cover $h : \bar{Y_\eta} \to Y_\eta$. We get the following commutative diagram

$$\begin{tikzcd}
\bar{Y_\eta} \arrow[r, "\psi"] \arrow[d, "h"'] & \pi_0(\bar{Y_\eta}) \arrow[d, "\pi_0(h)"] \\
Y_\eta \arrow[r, "\phi"] \arrow[d, "g"'] & \pi_0(Y_\eta) \arrow[d, "\pi_0(g)"] \\
Y \arrow[r, "\pi"] \arrow[d, "f"'] & \pi_0(Y) \\
X &
\end{tikzcd}$$

Each connected component of Y is a strict henselisation of X. Therefore it contains a point lying over a point of X_η, i.e. it contains a point in Y_η. This shows that $\pi_0(g)$ is surjective. Now $\pi_0(h)$ is clearly surjective, and we write

$$\alpha : \pi_0(\bar{Y_\eta}) \to \pi_0(Y)$$

for the composite. Now Y is w-contractible, so $\pi_0(Y)$ is extremally disconnected. It is therefore enough to show that $\alpha^*\mathcal{G}$ is locally constant. As endofunctors of $\mathrm{Sh}(\pi_0(\bar{Y_\eta})_{\text{ét}})$ we have

$$\psi_*\psi^* = id$$

so it is enough to show that $\psi^*\alpha^*\mathcal{G}$ is locally constant. But by the commutativity of the diagram, this is precisely $\mathcal{F}|_{\bar{Y_\eta}}$. Now $\bar{Y_\eta}$ is a w-contractible cover of X_η so by 7.53 it is locally constant. $\square$

We want to tie these notions to infinite Galois categories and so we need to find a natural fibre functor. Analogously to the definition of the étale fundamental group we fix a geometric point $\overline{x} : \operatorname{Spec}\Omega \to X$ and consider the functor of taking stalks.

$$\begin{aligned} ev_{\overline{x}} : \mathrm{Loc}(X_{\text{proét}}) &\to \mathbf{Set} \\ \mathcal{F} &\mapsto \mathcal{F}_{\overline{x}} \end{aligned}$$

We need the following lemma in the proof that $\mathrm{Loc}(X_{\text{proét}})$ is an infinite Galois category.

Lemma 7.55. *Let X be a connected locally Noetherian scheme. Let $x, y \in X$ be points. Then there exists a chain of points*

$$x = z_0, \ldots, z_n = y$$

such that z_{i+1} can be obtained from z_i by either specialization or generization.

Proof. Let U_x be the set of all points that can be reached by such a finite chain from x. Let W be the compliment $W_x = X \setminus U_x$. Let $z \in U_x$. Since X is locally Noetherian, there exists an open Noetherian neighborhood V of z. Clearly every point in V can reach z in a finite number of steps, since there are only finitely many connected components of V, so every point in V is reachable from x in finitely many steps, i.e. $V \subseteq U_x$. This shows that U_x is open. Now in the same manner we see that if $z \in W_x$ then any open Noetherian neighborhood of W_x lies entirely inside W_x, and so W_x is open. Since X is connected, $U_x \cup W_x = X$ and U_x and W_x are disoint, we have that one of them must be empty. By construction U_x is not empty, so we have $W_x = \emptyset$ and $U_x = X$. $\square$

Proposition 7.56. *The category* $\mathrm{Loc}(X_{\text{proét}})$ *of locally constant pro-étale sheaves on* X *is an infinite Galois category with a fibre functor* $ev_{\overline{x}}$

Proof. We go through the conditions one by one. We use the equivalence from 7.54 liberally and choose the most appropriate description of a local system for each condition to work with.

(1) We refer to Bhatt and Scholze [2015, §3.2 and Remark 7.3.4] to show that the inclusion $w\mathrm{Loc}(X_{\text{proét}}) \subseteq \mathrm{Sh}(X_{\text{proét}})$ commutes with all colimits and finite limits so $w\mathrm{Loc}(X_{\text{proét}})$ is closed under all colimits and finite limits since $\mathrm{Sh}(X_{\text{proét}})$ is.

(2) Here we use the equivalence $\mathrm{Loc}(X_{\text{proét}}) \cong \mathrm{Cov}_X$ and work with geometric coverings. We claim that the connected objects of $\mathrm{Cov}(X_{\text{proét}})$ are the $Y \in \mathrm{Cov}_X$ that are connected as schemes. First of all, any $Y \in \mathrm{Cov}(X_{\text{proét}})$ is locally Noetherian since X is. Assume that $Y \in \mathrm{Cov}(X_{\text{proét}})$ is connected as a scheme and consider a map $Z \to Y$ in $\mathrm{Cov}(X_{\text{proét}})$. The image of Z in Y is open, and it is closed under specializations by the valuative criteron of properness. As Y is locally Noetherian, open subsets are locally constructible. In general, any subset that is locally constructible and closed under specializations is closed. Thus the image of Z in Y is clopen and so empty or all of Y since Y is a connected scheme.

Now any scheme $Y \in \mathrm{Cov}(X_{\text{proét}})$ can be written as

$$Y = \coprod_i Y_i$$

where the Y_i's are connected X-schemes. What remains to be seen is that the Y_i's lie in $\mathrm{Cov}(X_{\text{proét}})$. The inclusion

$$Y_i \hookrightarrow Y$$

is an open immersion so the composition

$$Y_i \hookrightarrow Y \to X$$

is étale. Furthermore since Y is locally Noetherian, the connected components are clopen in Y. Thus the open immersion $Y_i \hookrightarrow Y$ is a flat closed immersion of finite presentation. In particular, it is proper so the composition $Y_i \hookrightarrow Y \to X$ is again proper and $Y_i \in \mathrm{Cov}_X$.

(3) We leave these set-theoretical considerations out.

(4) Since the fibre functor is the functor of taking stalks, it is a filtered colimit. It is a general categorical fact that filtered colimits commute with finite limits, see for example Stacks Project [2017, Tag 002W].

Stalks commute with colimits in general for sheaves on sites, see for example Stacks Project [2017, Tag 04EN].

To show that $ev_{\overline{x}}$ is faithful we assume that we are given morphisms $f, g : \mathcal{F} \to \mathcal{G}$ such that

$$ev_{\overline{x}}(f) = ev_{\overline{x}}(g)$$

By 7.55 we know that any point $y \in X$ can be reached by a finite number of specializations and generizations. Thus if z is a specialization or generization of x and we can show that $f_{\overline{x}} = g_{\overline{x}} \Rightarrow f_{\overline{z}} = g_{\overline{z}}$ for any geometric point $\overline{z}$ with set-theoretic image z, then we see that for any geometric point $\overline{y}$ with set theoretic image y we have $f_{\overline{y}} = g_{\overline{y}}$. Since equality of morphisms can be checked on stalks, this shows that $f = g$ and thus that the fiber functor is faithful.

So let $z \in \overline{\{x\}}$ and $\overline{z}$ a geometric point with set theoretic image z. Let $W \to X$ be an fpqc neighborhood of z. Then by definition, $W \to X$ is flat and for any affine open neighborhood U of x, there exists a quasi-compact $V \subseteq W$ that maps onto U. Since the sheaves we are working with are locally constant, we may choose W such that they are constant on W. But any open U that contains z contains x by construction so

$$f_{\overline{x}} = g_{\overline{x}} \Rightarrow f_{\overline{z}} = g_{\overline{z}}$$

Conversely, assume $x \in \overline{\{z\}}$. Then any open neighborhood of x contains z and we can use exactly the same argument to reduce to the case of constant sheaves.

(5) To show that $ev_{\overline{x}}$ is conservative, we use the same argument as we used to prove that it is faithful. Namely we assume that $f : \mathcal{F} \to \mathcal{G}$ is a morphism of locally constant sheaves and that $ev_{\overline{x}}(f) : ev_{\overline{x}}(\mathcal{F}) \to ev_{\overline{x}}(\mathcal{G})$ is an isomorphism of sets. Then we see, by reducing to the case of constant sheaves, that the induced map on stalks at $\overline{z}$ is an isomorphism for any specialization or generization of x, and therefore by 7.55 for any (geometric) point. Thus f is an isomorphism of sheaves.

(6) To prove that $ev_{\overline{x}}$ is tame, we fix a connected geometric cover $Y \to X$ and two geometric points $\overline{y}_1$ and $\overline{y}_2$ lying above $\overline{x}$. We need to show that there exists some $\gamma \in \pi_1(\mathrm{Loc}(X_{\text{proét}}), ev_{\overline{x}})$

such that $\gamma(\overline{y}_1) = \overline{y}_2$. Let y_1 and y_2 be the set-theoretic images of $\overline{y}_1$ and $\overline{y}_2$ respectively. Since Y is locally Noetherian, we can by 7.55 find a finite collection of points

$$y_1 = z_1, z_2, \ldots, z_n = y_2$$

such that z_{i+1} is either a generization or specialization of z_i for each i. Fix geometric points $\overline{z_i}$ with set-theoretic image z_i for each i. We obtain geometric points $\overline{x}_i$ in X with set-theoretic image x_i such that $\overline{z}_i$ lies above $\overline{x}_i$ for each i.

Now for each i we can choose a valuation ring R_i with an algebraically closed fraction field and a map

$$\mathrm{Spec}\,(R_i) \to Y$$

such that the special and generic points are sent isomorphically to $\overline{z_i}$ and $\overline{z_{i+1}}$ or vice versa (depending on whether z_{i+1} is a specialization or generization of z_i). We fix these rings and isomorphisms. These $\mathrm{Spec}\,(R_i) \to Y$ induce morphisms $\mathrm{Spec}\,(R_i) \to X$ which induce isomorphisms of fibre functors $ev_{\overline{x}_i} \cong ev_{\overline{x}_{i+1}}$ and so we get a chain of isomorphisms

$$ev_{\overline{x}} = ev_{\overline{x}_1} \cong ev_{\overline{x}_2} \cong \ldots \cong ev_{\overline{x}_n} = ev_{\overline{x}}$$

that is, we obtain an automorphism $\gamma \in \pi_1(\mathrm{Loc}(X_{\text{proét}}), ev_{\overline{x}})$. This is exactly the automorphism were looking for, since by construction

$$\gamma(\overline{y}_1) = \overline{y}_2$$

□

This allows us to define the *pro-étale fundamental* group of a scheme X with a base point $\overline{x}$ as the fundamental group of the infinite Galois category $\mathrm{Loc}(X_{\text{proét}})$

Definition 7.57. Let X be a scheme and fix a geometric point $\overline{x} : \mathrm{Spec}\,\Omega \to X$. We define the pro-étale fundamental group of X with basepoint $\overline{x}$ as the fundamental group of the infinite Galois category $\mathrm{Loc}(X_{\text{proét}})$ with fibre functor $ev_{\overline{x}}$, i.e.

$$\pi_1^{\text{proét}}(X, \overline{x}) := \pi_1(\mathrm{Loc}(X_{\text{proét}}), ev_{\overline{x}})$$

Recall that any open subgroup $U \leq \pi_1^{\text{proét}}(X, \overline{x})$ gives us a connected object in $\pi_1^{\text{proét}} - \mathbf{Set}$, namely G/U with stabilizer U for any element in G/U. Bacause of the equivalence of categories

$$\mathrm{Cov}(X_{\text{proét}}) \cong \mathrm{Loc}(X_{\text{proét}}) \cong \pi_1^{\text{proét}} - \mathbf{Set}$$

we have for each such open subgroup U a canonically defined covering $X_U \in \mathrm{Cov}(X_{\text{proét}})$ with a lift of basepoint to $\overline{y} \in X_U$ such that the structure map $X_U \to X$ preserves basepoints.

Furthermore, since $X_U \in \mathrm{Cov}(X_{\text{proét}})$ it is locally Noetherian so

$$\pi_1^{\text{proét}}(X_U, \overline{x}) = U$$

as subgroups of $\pi_1^{\text{proét}}(X, x)$.

We want to compare the étale fundamental groups and the pro-étale fundamental groups. We first show that the étale fundamental group is the profinite completion of the pro-étale fundamental group. Before we can show that we two lemmas.

Lemma 7.58. *Let X be a scheme and G a finite group. Then the G-torsors in $X_{\text{ét}}$ and the G-torsors in $X_{\text{proét}}$ are the same, i.e.*

$$(BG)(X_{\text{ét}}) \cong (BG)(X_{\text{proét}}).$$

Proof. Let $Y \to X$ be a G-torsor. Then there exists an étale cover $X' \to X$ that trivializes Y in the sense that the base change of $Y \to X$ by $X' \to X$ gives the trivial torsor on X';

$$\begin{tikzcd} X' \times G \arrow[r] \arrow[d] & Y \arrow[d] \\ X' \arrow[r] & X \end{tikzcd}$$

But since the map $Y \to X$ is étale, it is also weakly-étale, and since $X' \to X$ is étale it is fpqc so $Y \to X$ is a G-torsor in $X_{\text{proét}}$.

For the other direction, assume we have a pro-étale torsor $Y \to X$. Then there exists an fpqc map $X' \to X$ that trivializes it, i.e. such that the following square is cartesian.

$$\begin{tikzcd} X' \times G \arrow[r] \arrow[d] & Y \arrow[d] \\ X' \arrow[r] & X \end{tikzcd}$$

But then by fpqc descent, we can deduce that the map $Y \to X$ is étale. To show that it is actually a G-torsor in $X_{\text{ét}}$ we have to find an étale cover that trivializes it. But recall that an equivalent way of defining a

G-torsor on a site over X is as a scheme Y over X with a (right) action of G such that the obvious map

$$\begin{aligned} Y \times G &\to Y \times_X Y \\ (y, g) &\mapsto (y, yg) \end{aligned}$$

is an isomorphism of schemes. Now since Y is by assumption a G-torsor in $X_{\text{proét}}$, this map is an isomorphism and therefore Y is a G-torsor in $X_{\text{ét}}$ since $Y \to X$ is an étale map. $\square$

Lemma 7.59. *Let X be a scheme and $\overline{x}$ a geometric point in X. Let G be a finite group. Then there is an equivalence*

$$\underline{\mathrm{Hom}}(\pi_1^{\text{proét}}(X, \overline{x}), G) \cong (BG)(X_{\text{proét}})$$

where $\underline{\mathrm{Hom}}(H, G)$ denotes, for topological groups H and G, the groupoid of continuous maps $H \to G$ where maps between $f_1 : H \to G$ and $f_2 : H \to G$ are given by elements $g \in G$ conjugating f_1 into f_2

Proof. A map in $\underline{\mathrm{Hom}}(\pi_1^{\text{proét}}(X, \overline{x}), G)$ endows G with the structure of a $\pi_1^{\text{proét}}(X, \overline{x})$-set and thus corresponds to a geometric covering Y with automorphism group G. Furthemore it is clear that the multiplication action of G on the $\pi_1^{\text{proét}}(X, \overline{x})$-set G is free and transitive, and therefore so is the action of G on Y. Therefore $Y \to X$ is finite, and hence a Galois finite étale cover with automorphism group G. These correspond to G-torsors in $X_{\text{ét}}$ which are the same as G-torsors in $X_{\text{proét}}$ by 7.58.

On the other hand, let a G-torsor Y in $X_{\text{proét}}$ be given. It is then by 7.58 also a G-torsor in $X_{\text{proét}}$. That means that not only is $Y \to X$ (weakly-)étale it is finite. Finite morphisms are proper so $Y \to X$ is a geometric covering with automorphism group G and so corresponds to a continuous group homomorphism $\pi_1^{\text{proét}}(X, \overline{x}) \to G$. $\square$

Lemma 7.60. *Let X be a scheme, $\overline{x}$ be a geometric point in X and G be a profinite group. Then*

$$\underline{\mathrm{Hom}}(\pi_1^{\text{proét}}(X, \overline{x}), G) \cong (B\mathcal{F}_G)(X_{\text{proét}})$$

Proof. Both sides are compatible with cofiltered limits in G and so we can reduce to the case where G is finite, which is precicely 7.59 $\square$

Proposition 7.61. *Let X be a scheme and let $\overline{x} : \operatorname{Spec}\Omega \to X$ be a geometric point in X. Then the profinite completion of $\pi_1^{\text{proét}}(X, \overline{x})$ is $\pi_1^{\text{ét}}(X, \overline{x})$:*

$$\widehat{\pi_1^{\text{proét}}}(X, \overline{x}) \cong \pi_1^{\text{ét}}(X, \overline{x})$$

Proof. First we notice that if H is a group and we set $I_H = \cup S\underline{\mathrm{Hom}}(H, G)$ where the union is over all isomorphism classes of finite groups, and $S\underline{\mathrm{Hom}}(H, G)$ denotes the groupoid of all surjective, continuous group homomorphisms $H \to G$, then

$$\varprojlim_{i \in I_H} G_i = \hat{H}$$

where G_i denotes the target of the morphism i. Now let

$$I_{\text{ét}} = \bigcup S\underline{\mathrm{Hom}}(\pi_1^{\text{ét}}(X, \overline{x}), G)$$

and

$$I_{\text{proét}} = \bigcup S\underline{\mathrm{Hom}}(\pi_1^{\text{proét}}(X, \overline{x}))$$

where as before the union is taken over all isomorphism classes of finite groups. Now by 7.59 and 7.58 we have that

$$S\underline{\mathrm{Hom}}(\pi_1^{\text{ét}}(X, \overline{x}), G) = S\underline{\mathrm{Hom}}(\pi_1^{\text{proét}}(X, \overline{x}), G)$$

for all finite groups G. Therefore we have

$$\begin{aligned}
\pi_1^{\text{ét}}(X, \overline{x}) &= \pi_1^{\text{ét}}\widehat{(X, \overline{x})} \\
&= \varprojlim_{i \in I_{\text{ét}}} G_i \\
&= \varprojlim_{i \in I_{\text{proét}}} G_i \\
&= \pi_1^{\text{proét}}\widehat{(X, \overline{x})}
\end{aligned}$$

□

Furthermore we have the following proposition

Proposition 7.62. *If X is a scheme, $\overline{x} : \operatorname{Spec}\Omega \to X$ is a geometric point and if X is geometrically unibranch, then the étale and the pro-étale fundamental groups agree.*

$$\pi_1^{\text{proét}}(X, \overline{x}) = \pi_1^{\text{ét}}(X, \overline{x})$$

Proof. First we want to show that X is irreducible. We recall our assumption that X is connected, and so if the irreducible components of X are clopen, then this will immediately follow. Irreducible components are always closed, so it's the openness we need to look at.

Openness is a local property, so we may assume that X is Noetherian and since the irreducible components in a Noetherian topological space are finitely many, they are open.

The same argument shows that any connected $Y \in \mathrm{Cov}(X_{\text{proét}})$ is also irreducible.

Now let $\eta \in X$ be the generic point, let $Y \in \mathrm{Cov}(X_{\text{proét}})$ be a connected geometric covering and $Y_\eta \to \eta$ be the generic fibre. Since X is irreducible, the irreducible components of Y_η are in 1–1 correspondence with the irreducible components of Y that meet Y_η (see for example Grothendieck and Dieudonné [1971, §2.1.8]) and since Y is irreducible this shows that Y_η is irreducible and connected.

So we have that $Y_\eta \to \eta$ is a connected scheme in $\mathrm{Loc}(\eta_{\text{proét}})$. Now by 7.53 we see that $\mathrm{Loc}(\eta_{\text{proét}})$ is the category of disjoint unions of finite étale covers of η and so $Y_\eta \to \eta$ is finite étale. In particular we see that $\pi : Y \to X$ has finite fibres. As $\pi_1^{\text{ét}}(X, \overline{x})$ classifies finite étale covers of X and $\pi_1^{\text{proét}}(X, \overline{x})$ classifies geometric coverings of X, the statement of the proposition follows if we can show that $\pi : Y \to X$ is finite étale.

The property of being finite étale is local on the target, so we may assume that X is quasi-compact. Any quasi-compact $U \subseteq Y$ containing Y_η is finite étale over a quasi-compact open V and hence contains all fibres of points from V. We now expand U to U_1 contain the fibre over some point in the complement of V and obtain a quasi-compact open that it finite étale over some quasi-compact open $V_1 \subseteq X$ with $V \subsetneq V_1$. We continue like this to obtain a sequence of quasi-compact open subsets of X

$$V \subset V_1 \subset V_2 \dots$$

The Noetherianness of X then implies that for some $r \geq 0$ we have $V_r = X$, and since each corresponding $U_i \subseteq Y$ contains all fibres over V_i we see that $U_r = Y$, from which we obtain that $\pi : Y \to X$ is finite étale. □

Given a topological group G, or more generally a topological space, we define a sheaf associated with it. The following is Bhatt and Scholze [2015, Lem. 4.2.12].

Lemma 7.63. *Let X be a scheme and G a topological group. The association mapping any $U \in X_{\text{proét}}$ to $Map_{cont}(U, G)$ is a sheaf of groups on $X_{\text{proét}}$. We denote this sheaf by $\mathcal{F}_G$. If G is discrete, then $\mathcal{F}_G$ is the constant sheaf associated with G.*

Example 7.64. Consider again the nodal curve X obtained from $\mathbb{P}^1_k$ (where k is algebraically closed) by gluing 0 and ∞, and consider a geometric point $\overline{x} : \operatorname{Spec} k \to X$ over the node x. We have seen that all finite étale covers have the form Y_n where Y_n is n copies of $\mathbb{P}^1$ glued cyclically together (∞ of the i-th copy glues to the zero of the $i+1$-th copy etc.). However we also have a geometric covering Y_∞ that is obtained in the same manner, except that we have an infinite number of $\mathbb{P}^1$'s. This map is étale, but not finite. These are all the geometric coverings and all the groups $ev_{\overline{x}}(Y_n)$ are quotients of $ev_{\overline{x}}(Y_\infty)$ and so we see that

$$\pi_1^{\text{proét}}(X, \overline{x}) \cong \mathbb{Z}$$

Lemma 7.65. *Let X be a scheme and $\overline{x} : \operatorname{Spec} \Omega \to X$ be a geometric point. Then there is an equivalence of categories*

$$\operatorname{Loc}_{\mathbb{Z}_l}(X_{\text{proét}}) \cong \pi_1^{\text{proét}}(X, \overline{x}) - \operatorname{Rep_c}^{\mathbb{Z}_l}$$

Proof. Notice that for each n we have that $GL_n(\mathbb{Z}_l)$ is profinite and that the rank n locally constant sheaves of free $\mathbb{Z}_l$-modules on X are equivalent to $\mathcal{F}_{GL_n(\mathbb{Z}_l)}$-torsors and therefore to n-dimensional continuous representations $\pi_1^{\text{proét}}(X, \overline{x}) \to GL_n(\mathbb{Z}_l)$ by 7.60. ☐

Now we can present the main theorem. This is Bhatt and Scholze [2015, Lem. 7.4.7]

Theorem 7.66. *Let X be a scheme and $\overline{x} : \operatorname{Spec} \Omega \to X$ be a geometric point. Then there is an equivalence of categories*

$$\operatorname{Loc}_{\mathbb{Q}_l}(X_{\text{proét}}) \cong \pi_1^{\text{proét}}(X, \overline{x}) - \operatorname{Rep_c}^{\mathbb{Q}_l}$$

Proof. Let a continuous representation

$$\rho : \pi_1^{\text{proét}}(X, \overline{x}) \to GL_n(\mathbb{Q}_l)$$

be given for some n. Consider the subgroup $GL_n(\mathbb{Z}_l) \subseteq GL_n(\mathbb{Q}_l)$ and look at the preimage

$$U := \rho^{-1}(GL_n(\mathbb{Z}_l))$$

It is an open subgroup of $\pi_1^{\text{proét}}(X, x)$ and so it canonically defines a pointed geometric covering $X_U \to X$ with $\pi_1^{\text{proét}}(X_U, \overline{y}) = U$. The induced representation

$$\pi_1^{\text{proét}}(X_U, \overline{y}) \to GL_n(\mathbb{Z}_l)$$

defines by 7.65 a unique $\mathbb{Z}_l$-local system $M \in \operatorname{Loc}_{\mathbb{Z}_l}(X_{U,\text{proét}})$ and by tensoring with $\mathbb{Q}_l$ it defines a $\mathbb{Q}_l$-local system $M' \in \operatorname{Loc}_{\mathbb{Q}_l}(X_{U,\text{proét}})$. By descent, this M' defines a unique $N(\rho) \in \operatorname{Loc}_{\mathbb{Q}_l}(X_{\text{proét}})$.

On the other hand, if N is a $\mathbb{Q}_l$-local system, then it is a locally constant sheaf of $\mathbb{Q}_l$-vector spaces of finite rank, by definition. Therefore, there exists an $n \in \mathbb{N}$ such that we can view N as an $\mathfrak{F}_{GL_n(\mathbb{Q}_l)}$-torsor. For each $S \in GL_n(\mathbb{Q}_l) - \mathrm{Rep_c}$ we get an induced map

$$\rho_S : \mathfrak{F}_{GL_n(\mathbb{Q}_l)} \to \mathfrak{F}_{\mathrm{Aut}(S)}$$

By pushout of N along ρ_S we obtain a locally constant sheaf $N_S \in \mathrm{Loc}(X_{\text{proét}})$ with stalk S. This construction is functorial and we obtain a functor

$$\mathcal{G}_N : GL_n(\mathbb{Q}_l) - \mathrm{Rep_c} \to \mathrm{Loc}(X_{\text{proét}})$$

By 7.32 $GL_n(\mathbb{Q}_l)$ is Noohi and so $GL_n(\mathbb{Q}_l) - \mathbf{Set}$ is an infinite Galois gategory with the forgetful functor $For : GL_n(\mathbb{Q}_l) - \mathbf{Set} \to \mathbf{Set}$ as a fibre functor and Galois group $GL_n(\mathbb{Q}_l)$ by 7.37. Clearly this functor $\mathcal{G}_N$ is compatible with the fibre functors and so we obtain a continuous morphism of Galois groups in the opposite direction:

$$\rho_N : \pi_1^{\text{proét}}(X, x) \to GL_n(\mathbb{Q}_l)$$

$\square$

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